ENDS, SHAPES, AND BOUNDARIES IN MANIFOLD TOPOLOGY AND GEOMETRIC GROUP THEORY

C. R. GUILBAULT

ABSTRACT. This survey/expository article covers a variety of topics related to the "topology at infinity" of noncompact manifolds and complexes. In manifold topology and geometric group theory, the most important noncompact spaces are often contractible, so distinguishing one from another requires techniques beyond the standard tools of algebraic topology. One approach uses end invariants, such as the number of ends or the fundamental group at infinity. Another approach seeks nice compactifications, then analyzes the boundaries. A thread connecting the two approaches is shape theory.

In these notes we provide a careful development of several topics: homotopy and homology properties and invariants for ends of spaces, proper maps and homotopy equivalences, tameness conditions, shapes of ends, and various types of $\mathcal{Z}$-compactifications and $\mathcal{Z}$-boundaries. Classical and current research from both manifold topology and geometric group theory provide the context. Along the way, several open problems are encountered. Our primary goal is casual but coherent introduction that is accessible to graduate students and also of interest to active mathematicians whose research might benefit from knowledge of these topics.

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Preface

In [Sie70], a paper that plays a role in these notes, Siebenmann mused that his work was initiated at a time "when 'respectable' geometric topology was necessarily compact". That attitude has long since faded; today's topological landscape is filled with research in which noncompact spaces are the primary objects. Even so, past traditions have impacted today's topologists, many of whom developed their mathematical tastes when noncompactness was viewed more as a nuisance than an area for exploration. For that and other reasons, many useful ideas and techniques have been slow to enter the mainstream. One goal of this set of notes is to provide quick and intuitive access to some of those results and methods by weaving them together with more commonly used approaches, well-known examples, and current research. In this way, we attempt to present a coherent "theory of ends" that will be useful to mathematicians with a variety of interests.

Numerous topics included here are fundamental to manifold topology and geometric group theory: Whitehead and Davis manifolds, Stallings' characterization of Euclidean spaces, Siebenmann's Thesis, Chapman and Siebenmann's $\mathcal{Z}$-compactification Theorem, the Freudenthal-Hopf-Stallings Theorem on ends of groups, and applications of the Gromov boundary to group theory, are just a few. We hope these notes give the reader a better appreciation for some of that work. Many other results and ideas presented here are relatively new or still under development: generalizations of Siebenmann's thesis, Bestvina's $\mathcal{Z}$-structures on groups, use of $\mathcal{Z}$-boundaries in manifold topology, and applications of boundaries to non-hyperbolic groups, are among those discussed. There is much room for additional work on these topics; the natural path of our discussion will bring us near to a number of interesting open problems.

The style of these notes is to provide a lot of motivating examples. Key definitions are presented in a rigorous manner—often preceded by a non-rigorous, but (hopefully) intuitive introduction. Proofs or sketches of proofs are included for many of the fundamental results, while many others are left as exercises. We have not let issues of mathematical rigor prevent the discussion of important or interesting work. If a theorem or example is relevant, we try to include it, even when the proof is too long or deep for these pages. When possible, an outlines or key portions of an argument are provided—with implied encouragement for the reader to dig deeper.

These notes originated in a series of four one-hour lectures given at the workshop on *Geometrical methods in high-dimensional topology*, hosted by Ohio State University in the spring of 2010. Notes from those talks were expanded into a one-semester topics course at the University of Wisconsin-Milwaukee in the fall of that year. The author expresses his appreciation to workshop organizers Jean-Francois Lafont and Ian Leary for the opportunity to speak, and acknowledges all fellow participants in the OSU workshop and the UWM graduate students in the follow-up course—their feedback and encouragement were invaluable. Finally, thanks to my son Phillip Guilbault who created most of the figures in this document.

1. Introduction

A fundamental concept in the study of noncompact spaces is the "number of ends". For example, the line has two ends, the plane has one end, and the uniformly trivalent tree $\mathbb{T}_3$ has infinitely many ends. Counting ends has proven remarkably useful, but certainly there is more—after all, there is a qualitative difference between the single end of the ray $[0, \infty)$ and that of $\mathbb{R}^2$. This provides an idea: If, in the topological tradition of counting things, one can (somehow) use the π_0- or H_0-functors to measure the number of ends, then maybe the π_1- and H_1-functors (or, for that matter π_k and H_k), can be used in a similar manner to measure other properties of those ends. Turning that ideas into actual mathematics—the "end invariants" of a space—then using those invariants to solve real problems, is one focus of the early portions of these notes.

Another approach to confronting noncompact spaces is to compactify.[1] The 1-point compactification of $\mathbb{R}^1$ is a circle and the 1-point compactification of $\mathbb{R}^2$ a 2-sphere. A "better" compactification of $\mathbb{R}^1$ adds one point to each end, to obtain a closed interval—a space that resembles the line far more than does the circle. This is a special case of "end-point compactification", whereby a single point is added to each end of a space. Under that procedure, an entire Cantor set is added to $\mathbb{T}_3$, resulting in a compact, but still tree-like object. Unfortunately, the end-point compactification of $\mathbb{R}^2$ again yields a 2-sphere. From the point of view of preserving fundamental properties, a far better compactification of $\mathbb{R}^2$ adds an entire circle at infinity. This is a prototypical "$\mathcal{Z}$-compactification", with the circle as the "$\mathcal{Z}$-boundary". (The end-point compactifications of $\mathbb{R}^1$ and $\mathbb{T}_3$ are also $\mathcal{Z}$-compactifications.) The topic of $\mathcal{Z}$-compactification and $\mathcal{Z}$-boundaries is a central theme in the latter half of these notes.

Shape theory is an area of topology developed for studying compact spaces with bad local properties, so it may seem odd that "shapes" is one of just three topics explicitly mentioned in the title of this article devoted to noncompact spaces with nice local properties. This is not a mistake! As it turns out, the tools of shape theory are easily adaptable to the study of ends—and the connection is not just a similarity in approaches. Frequently, the shape of an appropriately chosen compactum precisely captures the illusive "topology at the end of a space". In addition, shape theory plays a clarifying role by connecting end invariants hinted at in paragraph one of this introduction to the $\mathcal{Z}$-boundaries mentioned in paragraph two. To those who know just enough about shape theory to judge it too messy and set-theoretical for use in manifold topology or geometric group theory (a belief briefly shared by this author), patience is encouraged. At the level of generality required for our purposes, shape theory is actually quite elegant and geometric. In fact, very little set-theoretic topology is involved—instead spaces with bad properties are quickly replaced by simplicial and CW complexes, where techniques are clean and intuitive. A working knowledge of shape theory is one subgoal of these notes.

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[1]Despite our affinity for noncompact spaces, we are not opposed to the practice of compactification, provided it is done in a (geometrically) sensitive manner.

1.1. **Conventions and notation.** Throughout this article, all spaces are separable metric. A *compactum* is a compact space. We often restrict attention to *absolute neighborhood retracts* (or *ANRs*)—a particularly nice class of spaces, whose most notable property is local contractibility. In these notes, ANRs are required to be locally compact. Notable examples of ANRs are: manifolds, locally finite polyhedra, locally finite CW complexes, proper CAT(0) spaces[2], and Hilbert cube manifolds. Due to their unavoidable importance, a short appendix with precise definitions and fundamental results about ANRs has been included. Readers anxious get started can safely begin, by viewing "ANR" as a common label for the examples just mentioned. An *absolute retract* (or AR) is a contractible ANR, while an *ENR* [resp., *ER*] is a finite-dimensional ANR [resp., AR].

The unmodified term *manifold* means "finite-dimensional manifold". A manifold is *closed* if it is compact and has no boundary and *open* if it is noncompact with no boundary; if neither is specified, boundary is permitted. For convenience, all manifolds are assumed to be piecewise-linear (PL); in other words, they may be viewed as simplicial complexes in which all links are PL homeomorphic to spheres of appropriate dimensions. A primary application of PL topology will be the casual use of general position and regular neighborhoods. A good source for that material is [RoSa82]. Nearly all that we do can be accomplished for smooth or topological manifolds as well; readers with expertise in those categories will have little trouble making the necessary adjustments.

Hilbert cube manifolds are entirely different objects. The *Hilbert cube* is the countably infinite product $\mathcal{Q} = \prod_{i=1}^{\infty} [-1, 1]$; X is a *Hilbert cube manifold* if each $x \in X$ has a neighborhood homeomorphic to $\mathcal{Q}$. Like ANRs, Hilbert cube manifolds play an unavoidably key role in portions of these notes. For that reason, we have included a short and simple appendix on Hilbert cube manifolds.

Symbols will be used as follows: $\approx$ denotes homeomorphism, while $\simeq$ indicates homotopic maps or homotopy equivalent spaces; $\cong$ indicates isomorphism. When M^n is a manifold, n indicates its dimension and ∂M^n its manifold boundary. When A is a subspace of X, $\mathrm{Bd}_X\, A$ (or when no confusion can arise, $\mathrm{Bd}\, X$) denote the set-theoretic boundary of A. The symbols $\overline{A}$ and $\mathrm{cl}_X\, A$ (or just $\mathrm{cl}\, A$) denote the closure of A in X, while $\mathrm{int}_X\, A$ (or just $\mathrm{int}\, A$) denotes the interior. The symbol $\widetilde{X}$ always denotes the universal cover of X. Arrows denote (continuous) maps or homomorphisms, with $\hookrightarrow$, $\rightarrowtail$, and $\twoheadrightarrow$ indicating inclusion, injection and surjection, respectively.

2. Motivating examples: contractible open manifolds

Let us assume that space-time is a large boundaryless 4-dimensional manifold. Recent evidence suggests that this manifold is noncompact (an "open universe"). By

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[2]A *proper* metric space is one in which every closed metric ball is compact.
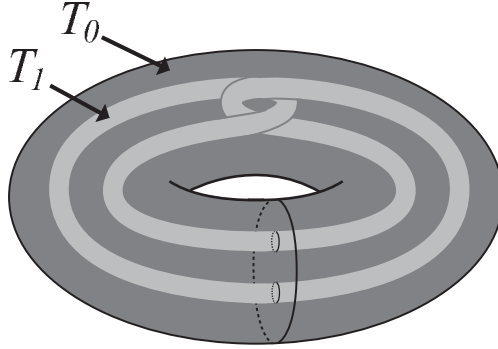
FIGURE 1. Constructing the Whitehead manifold

running time backward to the Big Bang, we might reasonably conclude that space-time is "just" a contractible open manifold[3]. Compared to the possibilities presented by a closed universe ($\mathbb{S}^4$, $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{R}P^4$, $\mathbb{C}P^2$, the E_8 manifold, $\cdots$?), the idea of a contractible open universe seems rather disappointing, especially to a topologist primed for the ultimate example on which to employ his/her tools. But there is a mistake in this thinking—an implicit assumption that a contractible open manifold is topologically uninteresting (no doubt just a blob, homeomorphic to an open ball). In this section we take a quick look at the surprisingly rich world of contractible open manifolds.

2.1. **Classic examples of exotic contractible open manifolds.** J.H.C White-head was among the first to underestimate contractible open manifolds. In an attempt to prove the Poincaré Conjecture, he briefly claimed that, in dimension 3, each is homeomorphic to $\mathbb{R}^3$. In [Whi35] he corrected that error by constructing the now famous *Whitehead contractible 3-manifold*—an object surprisingly easy to describe.

Example 1 (Whitehead's contractible open 3-manifold). *Let $\mathcal{W}^3 = \mathbb{S}^3 - T_\infty$, where T_∞ is the compact set (the **Whitehead continuum**) obtained by intersecting a nested sequence $T_0 \supseteq T_1 \supseteq T_2 \supseteq \cdots$ of solid tori, where each T_{i+1} is embedded in T_i in the same way that T_1 is embedded in T_0. See Figure 1. Standard tools of algebraic topology show that $\mathcal{W}^3$ is contractible. For example, first show that $\mathcal{W}^3$ is simply connected (this takes some thought). then show that it is acyclic with respect to $\mathbb{Z}$-homology.*

The most interesting question about $\mathcal{W}^3$ is: *Why is it not homeomorphic to $\mathbb{R}^3$?* Standard algebraic invariants are of little use, since $\mathcal{W}^3$ has the homotopy type of a point. But a variation on the fundamental group—the "fundamental group at infinity"—does the trick. Before developing that notion precisely, we describe a few more examples of *exotic* contractible open manifolds, ie, contractible open manifolds not homeomorphic to a Euclidean space.

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[3]No expertise in cosmology is being claimed by the author. This description of space-time is intended only to motivate discussion.

It turns out that exotic examples are rather common; moreover, they play important roles in both manifold topology and geometric group theory. But for now, let us just think of them as possible universes.

In dimension ≤ 2 there are no exotic contractible open manifolds, but in dimension 3, McMillan [Mc62] constructed uncountably many. In some sense, his examples are all variations on the Whitehead manifold. Rather than examining those examples, let us move to higher dimensions, where new possibilities emerge.

For $n \geq 4$, there exist *compact* contractible n-manifolds not homeomorphic to $\mathbb{B}^n$ (*exotic compact contractible manifolds*). Taking interiors provides a treasure trove of easy-to-understand exotic contractible open manifolds. We provide a simple construction for some of these objects

Example 2 (Newman contractible manifolds)**.** *Let G be a perfect group admitting a finite presentation with an equal number of generators and relators. The corresponding presentation 2-complex, K_G has the homology of a point. Embed K_G in $\mathbb{S}^n$ ($n \geq 5$) and let N be a regular neighborhood of K_G. By general position, loops and disks may be pushed off K_G, so inclusion induces an isomorphism $\pi_1(\partial N) \cong \pi_1(N) \cong G$. By standard algebraic topology arguments ∂N has the $\mathbb{Z}$-homology of an $(n-1)$-sphere and $C^n = \mathbb{S}^n - \operatorname{int} N$ has the homology of a point. A second general position argument shows that C^n is simply connected, and thus contractible—but C^n it is clearly not a ball. A compact contractible manifold constructed in this manner is called a **Newman compact contractible manifold** and its interior an **open Newman manifold**.*

Exercise 2.1. *Verify the assertions made in the above example. Be prepared to use numerous tools from a first course in algebraic topology: duality, universal coefficients, the Hurewicz theorem and a theorem of Whitehead (to name a few).*

The Newman construction can also be applied to acyclic 3-complexes. From that observation, one can show that every finitely presented *superperfect group G* (that is, $H_i(G; \mathbb{Z}) = 0$ for $i = 1, 2$) can be realized as $\pi_1(\partial C^n)$ for some compact contractible n-manifold ($n \geq 7$). A related result [Ker69], [FrQu90] asserts that *every $(n-1)$-manifold with the homology of $\mathbb{S}^{n-1}$ bounds a compact contractible n-manifold.* For an elementary construction of 4-dimensional examples, see [Maz61].

Exercise 2.2. *By applying the various Poincaré Conjectures, show that a compact contractible n-manifold is topologically an n-ball if and only if its boundary is simply connected. (An additional nontrivial tool, the Generalized Schönflies Theorem, may also be helpful.)*

A place where open manifolds arise naturally, even in the study of closed manifolds, is as covering spaces. A place where contractible open manifolds arise naturally is as universal covers of *aspherical* manifolds[4]. Until 1982, the following was a major open problem:

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[4]A connected space X is *aspherical* if $\pi_k(X) = 0$ for all $k \geq 2$.
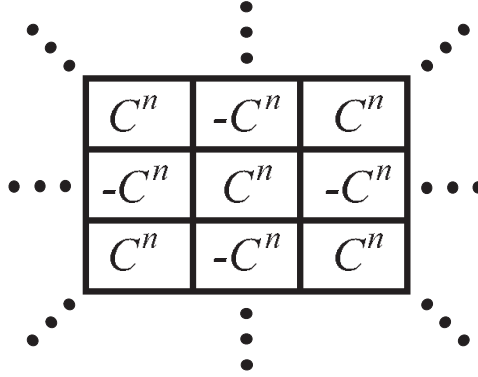
FIGURE 2. A Davis manifold

> *Does an exotic contractible open manifold ever cover a closed manifold?*
> *Equivalently: Can the universal cover of a closed aspherical manifold*
> *fail to be homeomorphic to $\mathbb{R}^n$?*

In dimension 3 this problem remained open until Perelman's solution to the Geometrization Conjecture. It is now known that the universal cover of a closed aspherical 3-manifold is always homeomorphic to $\mathbb{R}^3$. In all higher dimensions, a remarkable construction by Davis [Dvs83] produced aspherical n-manifolds with exotic universal covers.

Example 3 (Davis' exotic universal covering spaces). *The construction begins with an exotic (piecewise-linear) compact contractible oriented manifold C^n. Davis' key insight was that a certain Coxeter group Γ determined by a triangulation of ∂C^n provides precise instructions for assembling infinitely many copies of C^n into a contractible open n-manifold $\mathcal{D}^n$ with enough symmetry to admit a proper cocompact[5] action by Γ. Figure 2 provides a schematic, of $\mathcal{D}^n$, where $-C^n$ denotes a copy of C^n with reversed orientation. Intuitively, $\mathcal{D}^n$ is obtained by repeatedly reflecting copies of C^n across $(n-1)$-balls in ∂C^n. The reflections explain the reversed orientations on half of the copies. By Selberg's Lemma, there is a finite index torsion-free $\Gamma' \le \Gamma$. By properness, the action of Γ' on $\mathcal{D}^n$ is free (no $\gamma \in \Gamma'$ has a fixed point), so the quotient map $\mathcal{D}^n \to \Gamma' \backslash \mathcal{D}^n$ is a covering projection with image a closed aspherical manifold.*

Later, to prove that $\mathcal{D}^n \not\approx \mathbb{R}^n$, we will use an observation by Ancel and Siebenmann which, by ignoring all of the beautiful symmetry, provides a simple topological picture of $\mathcal{D}^n$. Let P^n and Q^n be oriented manifolds with connected boundaries, and let B, B' be $(n-1)$-balls in ∂P^n and ∂Q^n, respectively. A *boundary connected sum* $P^n \overset{\partial}{\#} Q^n$ is obtained by identifying B with B' via an orientation reversing homeomorphism. (By

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[5] An action by Γ on X is *proper* if, for each compact $K \subseteq X$ at most finitely many Γ-translates of K intersect K. The action is *cocompact* if there exists a compact C such that $\Gamma C = X$.
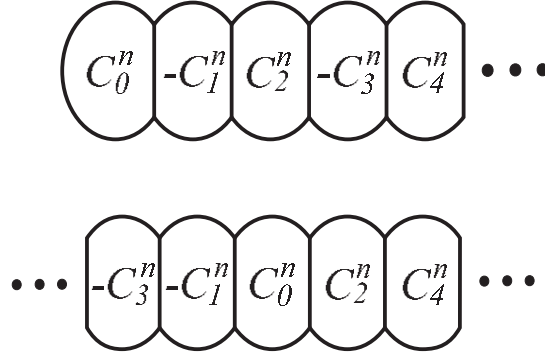
FIGURE 3. 1- and 2-ended boundary connected sums

using an orientation reversing gluing map, we may give $P^n \overset{\partial}{\#} Q^n$ an orientation that agrees with both original orientations.).

Theorem 2.3. [AnSi85] *A Davis manifold $\mathcal{D}^n$ constructed from copies of an oriented compact contractible manifold C^n is homeomorphic to the interior of an infinite boundary connected sum:*

$$C_0^n \overset{\partial}{\#} (-C_1^n) \overset{\partial}{\#} (C_2^n) \overset{\partial}{\#} (-C_3^n) \overset{\partial}{\#} \cdots$$

where each C_{2i}^n is a copy of C^n and each $-C_{2i+1}^n$ is a copy of $-C^n$.

Remark 1. The reader is warned that an infinite boundary connected sum is not topologically well-defined. For example, one could arrange that the result be 2-ended instead of 1-ended. See Figure 3. Remarkably, the *interior* of such a sum is well-defined. The proof of that fact is relatively straight-forward; it contains the essence of Theorem 2.3.

Exercise 2.4. *Sketch a proof that the 1-ended and 2-ended versions of $C_0^n \overset{\partial}{\#} (-C_1^n) \overset{\partial}{\#} (C_2^n) \overset{\partial}{\#} (-C_3^n) \overset{\partial}{\#} \cdots$, indicated by Figure 3 have homeomorphic interiors.*

Example 4 (Asymmetric Davis manifolds). *To create a larger collection of exotic contractible open n-manifolds (without concern for whether they are universal covers), the infinite boundary connect sum construction can be applied to a collection $\left\{ C_j^n \right\}_{j=0}^{\infty}$ of non-homeomorphic compact contractible n-manifolds. Here orientations are less relevant, so mention is omitted. Since there are infinitely many distinct compact contractible n-manifolds, this strategy produces uncountably many examples, which we refer to informally as **asymmetric Davis manifolds**. Distinguishing one from another will be a good test our soon-to-be-developed tools. Recent applications of these objects can be found in [Bel] and in the dissertation of P. Sparks.*

Exercise 2.5. *Show that the interior of an infinite boundary connected sum of compact contractible n-manifolds is contractible.*

A natural question is motivated by the above discussion:

> *Among the contractible open manifolds described above, which can or cannot be universal covers of closed n-manifolds?*

We will return to this question in §5.1. For now we settle for a fun observation by McMillan and Thickstun [McTh80].

Theorem 2.6. *For each $n \geq 3$, there exist exotic contractible open n-manifolds that are not universal covers of any closed n-manifold.*

Proof. There are uncountably many exotic open n-manifolds and, by [ChKi70], only countably many closed n-manifolds.   □

2.2. **Fundamental groups at infinity for the classic examples.** With an ample collection of examples, we begin defining an algebraic invariant useful for distinguishing one contractible open manifold from another. Technical issues will arise, but to keep focus on the big picture, we delay confronting those until later. Once complete, the new invariant will be widely applicable, but for now we concentrate on contractible open manifolds.

Let W^n be a contractible open manifold with $n \geq 2$. Express W^n as $\cup_{i=0}^{\infty} K_i$ where each K_i is a connected codimension 0 submanifold and $K_i \subseteq \text{int } K_{i+1}$ for each i. With some additional care, arrange that each K_i has connected complement. (Here one uses the fact that W^n is contractible and $n \geq 2$. See Exercise 3.2.) The corresponding *neighborhoods of infinity* are the sets $U_i = \overline{W^n - K_i}$.

For each i, let $p_i \in U_i$ and consider the inverse sequence of groups:

$$(2.1) \qquad \pi_1\left(U_0, p_0\right) \xleftarrow{\lambda_1} \pi_1\left(U_1, p_1\right) \xleftarrow{\lambda_2} \pi_1\left(U_2, p_2\right) \xleftarrow{\lambda_3} \cdots$$

We would *like* to think of the λ_i as being induced by inclusion, but since $\cap_{i=0}^{\infty} U_i = \varnothing$, a single choice of base point is impossible. Instead, for each i choose a path α_i in U_i connecting p_i to p_{i+1}; then declare λ_i to be the composition

$$\pi_1\left(U_{i-1}, p_{i-1}\right) \xleftarrow{\widehat{\alpha}_{i-1}} \pi_1\left(U_{i-1}, p_i\right) \leftarrow \pi_1\left(U_i, p_i\right)$$

where the first map is induced by inclusion and $\widehat{\alpha}_{i-1}$ is the "change of base point isomorphism". By assembling the α_i end-to-end, we can define a map $r : [0, \infty) \to X$, called the *base ray*. The entire inverse sequence (2.1) is taken as a representation of the *fundamental group at infinity (based at r)* of W^n. Those who prefer a single group can take an inverse limit (defined in §4.1) to obtain the *Čech fundamental group at infinity* (based at r). Unfortunately, that inverse limit typically contains far less information than the inverse sequence itself—more on that later.

Two primary technical issues are already evident:

- **well-definedness:** most obviously, the groups found in (2.1) depend upon the chosen neighborhoods of infinity, and
- **dependence upon base ray:** the "bonding homomorphisms" in (2.1) depend upon the base ray.

We will return to these issues soon; for now we forge ahead and apply the basic idea to some examples.

Example 5 (Fundamental group at infinity for $\mathbb{R}^n$). *Express $\mathbb{R}^n$ as $\cup_{i=0}^{\infty} i\mathbb{B}^n$ where $i\mathbb{B}^n$ is the closure of the ball of radius i. Then, $U_0 = \mathbb{R}^n$ and for $i > 0$, $U_i = \overline{\mathbb{R}^n - \mathbb{B}_i}$ is homeomorphic to $\mathbb{S}^{n-1} \times [i, \infty)$. If we let r be a true ray emanating from the origin and $p_i = r \cap (\mathbb{S}^{n-1} \times \{i\})$ we get a representation of the fundamental group at infinity as*

$$(2.2) \qquad\qquad 1 \leftarrow 1 \leftarrow 1 \leftarrow 1 \leftarrow \cdots$$

when $n \geq 3$, and when $n = 2$, we get (with a slight abuse of notation)

$$(2.3) \qquad\qquad 1 \leftarrow \mathbb{Z} \xleftarrow{\text{id}} \mathbb{Z} \xleftarrow{\text{id}} \mathbb{Z} \xleftarrow{\text{id}} \cdots$$

Modulo the technical issues, we have a modest application of the fundamental group at infinity—it distinguishes the plane from higher-dimensional Euclidean spaces.

Example 6 (Fundamental group at infinity for open Newman manifolds). *Let C^n be a compact contractible n-manifold and $G = \pi_1(\partial C^n)$. By deleting ∂C^n from a collar neighborhood of ∂C^n in C^n we obtain an* open collar *neighborhood of infinity $U_0 \approx \partial C^n \times [0, \infty)$ in the open manifold $\text{int}\, C^n$. For each $i \geq 1$, let U_i be the subcollar corresponding to $\partial C^n \times [i, \infty)$ and let r to be the ray $\{p\} \times [0, \infty)$, with $p_i = p \times \{i\}$. We get a representation of the fundamental group at infinity*

$$G \xleftarrow{\text{id}} G \xleftarrow{\text{id}} G \xleftarrow{\text{id}} \cdots$$

The (still-to-be-quantified) difference between this and (2.2) verifies that $\text{int}\, C^n$ is not homeomorphic to $\mathbb{R}^n$.

Example 7 (Fundamental group at infinity for Davis manifolds). *To aid in obtaining a representation of the fundamental group at infinity of a Davis manifold $\mathcal{D}^n$, we use Theorem 2.3 to view $\mathcal{D}^n$ as the interior of $C_0 \overset{\partial}{\#} (-C_1) \overset{\partial}{\#} (C_2) \overset{\partial}{\#} (-C_3) \overset{\partial}{\#} \cdots$, where each C_i is a copy of a fixed compact contractible n-manifold C. (Superscripts omitted to avoid excessive notation.)*

Borrow the setup from Example 6 to express $\text{int}\, C$ as $\cup_{i=0}^{\infty} K_i$ where each $K_i \doteq \overline{\text{int}\, C - U_i}$ is homeomorphic to C^n. We may exhaust $\mathcal{D}^n$ by compact contractible manifolds $L_i \approx C_0 \overset{\partial}{\#}(-C_1) \overset{\partial}{\#} \cdots \overset{\partial}{\#}(\pm C_i)$ created by "tubing together" $K_i^0 \cup (-K_i^1) \cup \cdots \cup (\pm K_i^i)$, where the tubes are copies of $\mathbb{B}^{n-1} \times [-1, 1]$ and K_i^j is the copy of K_i in $\pm C_j$. See Figure 4. It is easy to see that a corresponding neighborhood of infinity $V_i = \overline{\mathcal{D}^n - L_i}$ has fundamental group $G_0 * G_1 * \cdots * G_i$ where each G_i is a copy of G; moreover, the homomorphism of $G_0 * G_1 * \cdots * G_i * G_{i+1}$ to $G_0 * G_1 * \cdots * G_i$ induced by $V_{i+1} \hookrightarrow V_i$ acts as the identity on $G_0 * G_1 * \cdots * G_i$ and sends G_{i+1} to 1. With appropriate choices of base points and ray, we arrive at a representation of the fundamental group at infinity of $\mathcal{D}^n$ of the form

$$(2.4) \qquad G_0 \leftarrow G_0 * G_1 \leftarrow G_0 * G_1 * G_2 \leftarrow G_0 * G_1 * G_2 * G_3 \leftarrow \cdots.$$
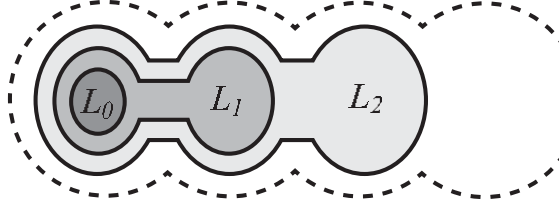
FIGURE 4. An exhaustion of $\mathcal{D}^n$ by compact contractible manifolds

Example 8 (Fundamental group at infinity for asymmetric Davis manifolds). *By proceeding as in Example 7, but not requiring $C_j \approx C_k$ for $j \neq k$, we obtain manifolds with fundamental groups at infinity represented by inverse sequences like (2.4), except that the various G_i need not be the same. By choosing different sequences of compact contractible manifolds, we can arrive at an uncountable collection of inverse sequences. Some work is still necessary in order to claim an uncountable collection of topologically distinct manifolds.*

Example 9 (Fundamental group at infinity for the Whitehead manifold). *For each $i \geq 0$, let $A_i = \overline{T_i - T_{i+1}}$. Then A_i is a compact 3-manifold, with a pair of torus boundary components ∂T_i and ∂T_{i+1}. Standard techniques from 3-manifold topology allow one to show that $G = \pi_1(A_i)$ is nonabelian and that each boundary component is incompressible in A_i, ie, $\pi_1(\partial T_i)$ and $\pi_1(\partial T_{i+1})$ inject into G. If we let A_{-1} be the solid torus $\overline{\mathbb{S}^3 - T_0}$, then $\mathcal{W}^3 = A_{-1} \cup A_0 \cup A_1 \cup A_2 \cup \cdots$, where $A_i \cap A_{i+1} = T_{i+1}$ for each i. Set $U_i = A_i \cup A_{i+1} \cup A_{i+2} \cup \cdots$ for each $i \geq 0$, to obtain a nested sequence of homeomorphic neighborhoods of infinity, each having fundamental group isomorphic to an infinite free product with amalgamation*

$$\pi_1(U_i) = G_i *_\Lambda G_{i+1} *_\Lambda G_{i+2} *_\Lambda \cdots.$$

where $\Lambda \cong \mathbb{Z} \oplus \mathbb{Z}$. Assembling these into an inverse sequence (temporarily ignoring base ray issues) gives a representation of the fundamental group at infinity

$$G_0 *_\Lambda G_1 *_\Lambda G_2 *_\Lambda G_4 *_\Lambda \cdots \longleftarrow G_1 *_\Lambda G_2 *_\Lambda G_3 *_\Lambda \cdots \longleftarrow G_2 *_\Lambda G_3 *_\Lambda \cdots \longleftarrow.$$

Combinatorial group theory provides a useful observation: each bonding homomorphism is injective and none is surjective.

We will return to the calculations from this section when enough mathematical rigor has been built up to make them truly useful.

3. BASIC NOTIONS IN THE STUDY OF NONCOMPACT SPACES

An important short-term goal is confront the issue of well-definedness and to clarify the role of the base ray in our above approach to the fundamental group at infinity. Until that is done, the calculations in the previous section should be viewed with some skepticism. Since we will eventually broaden our scope to spaces far more general than contractible open manifolds, we first take some time to lay out a variety general facts and definitions of use in the study of noncompact spaces.

3.1. **Neighborhoods of infinity and ends of spaces.** A subset U of a space X is a *neighborhood of infinity* if $\overline{X - U}$ is compact; a subset of X is *unbounded* if its closure is noncompact. (Note. This differs from the metric notion of "unboundedness", which is dependent upon the metric.) We say that X *has k ends*, if k is a least upper bound on the number of unbounded components in a neighborhood of infinity. If no such k exists, we call X infinite-ended.

Example 10. *The real line has 2 ends while, for all $n \geq 2$, $\mathbb{R}^n$ is 1-ended. A space is compact if and only if it is 0-ended. A common example of an infinite-ended space is the universal cover of $\mathbb{S}^1 \vee \mathbb{S}^1$.*

Exercise 3.1. *Show that an ANR X that admits a proper action by an infinite group G, necessarily has $1, 2$, or infinitely many ends. (This is a key ingredient in an important theorem from geometric group theory. See §6.)*

Exercise 3.2. *Show that a contractible open n-manifold of dimension ≥ 2 is always 1-ended.* **Hint.** *Ordinary singular or simplicial homology will suffice.*

An *exhaustion of X by compacta* is a nested sequence $K_0 \subseteq K_1 \subseteq K_2 \subseteq$ of compact subsets whose union is X; in this case the corresponding collection of neighborhoods of infinity $U_i = X - K_i$ is *cofinal*, ie, $\cap_{i=0}^{\infty} U_i = \varnothing$. A compactum K_i is *efficient* if it is connected and the corresponding U_i has only unbounded components. An exhaustion of X by efficient compacta is called an *efficient exhaustion*. The following is an elementary, but nontrivial, exercise in general topology.

Exercise 3.3. *Show that every connected ANR X admits an efficient exhaustion by compacta.* **Note.** *For this exercise, it suffices to replace the ANR hypothesis with the weaker assumption of locally compact and locally path connected.*

Let $\{K_i\}_{i=0}^{\infty}$ be a efficient exhaustion of X by compacta and, for each i, let $U_i = X - K_i$. Let $\mathcal{E}nds(X)$ be the set of all sequences $(V_0, V_1, V_2, \cdots)$ where V_i is a component of U_i and $V_0 \supseteq V_1 \supseteq \cdots$. Give $\overline{X} = X \cup \mathcal{E}nds(X)$ the topology generated by the basis consisting of all open subsets of X and all sets $\overline{V}_i$ where

$$\overline{V}_i = V_i \cup \{(W_0, W_1, \cdots) \in \mathcal{E}nds(X) \mid W_i = V_i\}.$$

Then $\overline{X}$ is separable, compact, and metrizable; it is known as the *Freudenthal compactification* of X.

Exercise 3.4. *Verify the assertions made in the final sentence of the above paragraph. Then show that any two efficient exhaustion of X by compacta result in compactifications that are canonically homeomorphic.*

Exercise 3.5. *Show that the cardinality of $\mathcal{E}nds(X)$ agrees with the "number of ends of X" defined at the beginning of this section.*

A *closed [open] neighborhood of infinity* in X is one that is closed [open] as a subset of X. If X is an ANR, we often prefer a neighborhoods of infinity to themselves be ANRs. This is automatic for open, but not for closed neighborhoods of infinity. Call

a neighborhood of infinity *sharp if it is closed and also an ANR.* Call a space X *sharp at infinity* if it contains arbitrarily small sharp neighborhoods of infinity, ie, if every neighborhood of infinity in X contains one that is sharp.

Example 11. *Manifolds, locally finite polyhedra, and finite-dimensional locally finite CW complexes are sharp at infinity—they contain arbitrarily small closed neighborhoods of infinity that are themselves manifolds with boundary, locally finite polyhedra, and locally finite CW complexes, respectively. In a similar manner, Hilbert cube manifolds are sharp at infinity by an application of Theorem B.2.*

Example 12. *Every proper CAT(0) spaces X is sharp at infinity—but this is not entirely obvious. The most natural type of closed neighborhood of infinity, $N_{p,r} = X - B(p; r)$, is an ANR if and only if the metric sphere $S(p; r)$ is an ANR. Surprisingly, it is not known whether this is always the case. But we can fatten $N_{p,r}$ to an ANR by applying Exercise A.5.*

Problem 1. *In a proper CAT(0) spaces X, is each $S(p; r)$ an ANR? Does there exist some $p_0 \in X$ and a sequence of arbitrarily large r_i for which each $S(p_0; r_i)$ is an ANR? Does it help to assume that X is finite-dimensional or that X is a manifold?*

An especially nice variety of sharp neighborhood is available in n-manifolds or a Hilbert cube manifolds. A closed neighborhood of infinity $N \subseteq M^n$ in an n-manifold with compact boundary is *clean* if it is a codimension 0 submanifold disjoint from ∂M^n and $\partial N = \mathrm{Bd}_{M^n} N$ has a product neighborhood ($\approx \partial N \times [0, 1]$) in M^n. In a Hilbert cube manifold X, where there is no intrinsic notion of boundary (recall that $\mathcal{Q}$ itself is homogeneous!), we simply require that $\mathrm{Bd}_X N$ be a Hilbert cube manifold with a product neighborhood in X. In an n-manifold with noncompact boundary a natural, but slightly more complicated, definition is possible; it is not needed in these notes.

3.2. **Proper maps and proper homotopy type.** A map $f : X \to Y$ is *proper*[6] if $f^{-1}(C)$ is compact for all compact $C \subseteq Y$.

Exercise 3.6. *Show that $f : X \to Y$ between locally compact metric spaces is proper if and only if the obvious extension to their 1-point compactifications is continuous.*

A pair of maps $f_0, f_1 : X \to Y$ are *properly homotopic* is there is a proper map $H : X \times [0, 1] \to Y$, with $H_0 = f_0$ and $H_1 = f_1$. We call H a *proper homotopy* between f_0 and f_1 and write $f_0 \overset{p}{\simeq} f_1$. We say that $f : X \to Y$ is a *proper homotopy equivalence* if there exists $g : Y \to X$ such that $gf \overset{p}{\simeq} \mathrm{id}_X$ and $fg \overset{p}{\simeq} Y$. In that case we say X and Y are *proper homotopy equivalent* and write $X \overset{p}{\simeq} Y$.

Remark 2. It is immediate that homeomorphisms are both proper maps and proper homotopy equivalences, but many pairs of spaces that are homotopy equivalent in the traditional sense are not *proper* homotopy equivalent. For example, whereas all

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[6]Yes, this is our third distct mathematical use of the word *proper*!

contractible open manifolds (indeed, all contractible spaces) are homotopy equivalent, they are frequently distinguished by their proper homotopy types.

It would be impossible to overstate the importance of "properness" in the study of noncompact spaces. Indeed, it is useful to think in terms of the *proper categories* where the objects are spaces (or certain subclasses of spaces) and the morphisms are proper maps or proper homotopy classes of maps. In the latter case, the isomorphisms are precisely the proper homotopy equivalences. Most of the invariants defined in these notes (such as the fundamental group at infinity) can be viewed as functors on the proper homotopy category of appropriate spaces.

The following offers a sampling of the usefulness of proper maps in understanding noncompact spaces.

Proposition 3.7. *Let $f : X \to Y$ be a proper map between ANRs. Then*

(1) *f induces a canonical function $f^* : \mathcal{E}\mathrm{nds}\,(X) \to \mathcal{E}\mathrm{nds}\,(Y)$ that may be used to extend f to a map $\overline{f} : \overline{X} \to \overline{Y}$ between Freudenthal compactifications,*
(2) *if $f_0, f_1 : X \to Y$ are properly homotopic, then $f_0^* = f_1^*$, and*
(3) *if $f : X \to Y$ is a proper homotopy equivalence, then f^* is a bijection.*

Proof. Begin with efficient exhaustions $\{K_i\}$ and $\{L_i\}$ of X and Y, respectively. The following simple observations make the uniqueness and well-definedness of f^* straight-forward:

(1) By properness, for each i, there is a k_i such that $f\,(X - K_{k_i}) \subseteq Y - L_i$,
(2) By connectedness, a given component U_i of $X - K_{k_i}$ is sent into a unique component V_i of $Y - L_i$,
(3) By nestedness, each entry W_j of $(W_0, W_1, \cdots) \in \mathcal{E}\mathit{nds}\,(X)$ determines all entries of lower index; hence every subsequence of entries determines that element.

$\square$

Exercise 3.8. *Fill in the remaining details in the proof of Proposition 3.7.*

The following observation is a key sources of proper maps and proper homotopy equivalences.

Proposition 3.9. *Let $f : X \to Y$ be a proper map between connected ANRs inducing an isomorphism on fundamental groups. Then the lift $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ to universal covers is a proper map. If $f : X \to Y$ be a proper homotopy equivalence, then $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ is a proper homotopy equivalence.*

Corollary 3.10. *If $f : X \to Y$ is a homotopy equivalence between compact connected ANRs, then $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ is a proper homotopy equivalence.*

We prove a simpler Lemma that leads directly to Corollary 3.10 and contains the ideas needed for Proposition 3.9. A different approach and more general results can be found in [Geo08, §10.1].

Lemma 3.11. *If $k : A \to B$ is a map between compact connected ANRs inducing an isomorphism on fundamental groups, then the lift $\widetilde{k} : \widetilde{A} \to \widetilde{B}$ between universal covers is proper.*

Proof of Lemma 3.11. Let G denote $\pi_1(A) \cong \pi_1(B)$. Then G acts by covering transformations (properly, cocompactly and freely) on $\widetilde{A}$ and $\widetilde{B}$ so that $\widetilde{k}$ is G-equivariant. Let $K \subseteq \widetilde{A}$ and $L \subseteq \widetilde{B}$ be compacta such that $GK = \widetilde{A}$ and $GL = \widetilde{B}$; without loss of generality, arrange that $G \cdot \text{int}(L) = \widetilde{B}$ and $\widetilde{k}(K) \subseteq L$. The assertion follows easily if $\widetilde{k}^{-1}(L)$ is compact. Suppose otherwise. Then there exists a sequence $\{g_i\}_{i=1}^{\infty}$ of distinct element of G for which $g_i K \cap \widetilde{k}^{-1}(L) \neq \varnothing$. But then each $g_i L$ intersects L, contradicting properness. $\square$

3.3. **Proper rays.** Henceforth, we refer to any proper map $r : [a, \infty) \to X$ as a *proper ray* in X. In particular, we do not require a proper ray to "straight" or even an embedding. A *reparameterization r'* of a proper ray r is obtained precomposing r with a homeomorphism $h : [b, \infty) \to [a, \infty)$. Note that a reparameterization of a proper ray is proper.

Exercise 3.12. *Show that the base ray $r : [0, \infty) \to X$ described in §2.2 is proper. Conversely, Let $s : [0, \infty) \to X$ be a proper ray, $\{K_i\}_{i=0}^{\infty}$ an efficient exhaustion of X by compacta, and for each i, $U_i = X - K_i$. Show that, by omitting an initial segment $[0, a)$ and then reparameterizing $s|_{[a,\infty)}$, we may obtain a corresponding proper ray $r : [0, \infty) \to X$ with $r([i, i+1]) \subseteq U_i$ for each i. In this way, any proper ray in X can be used as a base ray for a representation of the fundamental group at infinity.*

Declare proper rays $r, s : [0, \infty) \to X$ to be *strongly equivalent* if they are properly homotopic and *weakly equivalent* if there is a proper homotopy $K : \mathbb{N} \times [0, 1] \to X$ between $r|_{\mathbb{N}}$ and $s|_{\mathbb{N}}$. Equivalently, r and s are weakly equivalent if there is a proper map h of the *infinite ladder* $L_{[0,\infty)} = ([0, \infty) \times \{0, 1\}) \cup (\mathbb{N} \times [0, 1])$ into X, with $h|_{[0,\infty) \times 0} = r$ and $h|_{[0,\infty) \times 1} = s$. Properness ensures that rungs near the end of $L_{[0,\infty)}$ map toward the end of X. When the squares in the ladder can be filled in with a proper collection of 2-disks in X, a weak equivalence can be promoted to a strong equivalence.

For the set of all proper rays in X with domain $[0, \infty)$, let $\mathcal{E}(X)$ be the set of weak equivalence classes and $\mathcal{SE}(X)$ the collection of set of strong equivalence. There is an obvious surjection $\Phi : \mathcal{SE}(X) \to \mathcal{E}(X)$. We say that X is *connected at infinity* if $|\mathcal{E}(X)| = 1$ and *strongly connected at infinity* if $|\mathcal{SE}(X)| = 1$.

Exercise 3.13. *Show that, for ANRs, there is a one-to-one correspondence between $\mathcal{E}(X)$ and $\mathcal{E}nds(X)$. (Hence, proper rays provide an alternative, and more geometric, method for defining the ends of a space.)*

Exercise 3.14. *Show that $\Phi : \mathcal{SE}(L_{[0,\infty)}) \to \mathcal{E}(L_{[0,\infty)})$ is not injective. In fact $\mathcal{SE}(L_{[0,\infty)})$ is uncountable. (This is the prototypical example where $\mathcal{SE}(X)$ differs from $\mathcal{E}(X)$.)*

3.4. **Finite domination and homotopy type.** In addition to properness, there are notions related to homotopies and homotopy types that are of particular importance in the study of noncompact spaces. We introduce some of those here.

A space Y has *finite homotopy type* (or simply *finite type*) if it is homotopy equivalent to a finite CW complex; it is *finitely dominated* if there is a finite complex K and maps $u : Y \to K$ and $d : K \to Y$ such that $d \circ u \simeq \mathrm{id}_Y$. In this case, the map d is called a *domination* and we say that K dominates Y.

Proposition 3.15. *Suppose Y is finitely dominated with maps $u : Y \to K$ and $d : K \to Y$ satisfying the definition. Then*

(1) $H_k(Y; \mathbb{Z})$ *is finitely generated, for all k*
(2) $\pi_1(Y, y_0)$ *is finitely presentable, and*
(3) *if Y' is homotopy equivalent to Y, then Y' is finitely dominated.*

Proof. Since d induces surjections on all homology and homotopy groups, the finite generation of $H_k(Y; \mathbb{Z})$ and $\pi_1(Y, y_0)$ are immediate. The finite presentability of the latter requires some elementary combinatorial group theory; an argument (lifted from [Wal65]) can be found in [Gui00, Lemma 2]. The final item is let as an exercise. $\square$

Exercise 3.16. *Show that if Y' is homotopy equivalent to Y and Y if finitely dominated, then Y' is finitely dominated.*

The next proposition adds some intuitive meaning to finite domination.

Proposition 3.17. *An ANR Y is finitely dominated if and only if there exists a self-homotopy that "pulls Y into a compact subset", ie, $H : Y \times [0, 1] \to Y$ such that $H_0 = \mathrm{id}_Y$ and $\overline{H_1(Y)}$ is compact.*

Proof. If $u : Y \to K$ and $d : K \to Y$ satisfy the definition of finite domination, then the homotopy between id_Y and $d \circ u$ pulls Y into $d(K)$.

For the converse, begin by assuming that Y is a locally finite polyhedron. If $H : Y \times [0, 1] \to X$ such that $H_0 = \mathrm{id}_X$ and $\overline{H_1(Y)}$ is compact, then any compact polyhedral neighborhood K of $\overline{H_1(Y)}$ dominates Y, with $u = H_1$ and d the inclusion.

For the general case, we use some Hilbert cube manifold magic. By Theorem B.1, $Y \times \mathcal{Q}$ is a Hilbert cube manifold, so by Theorem B.2, $Y \times \mathcal{Q} \approx P \times \mathcal{Q}$, where P is a locally finite polyhedron. The homotopy that pulls Y into a compact set can be used to pull $P \times \mathcal{Q}$ into a compact subset of the form $K \times \mathcal{Q}$, where K is a compact polyhedron. It follows easily that K dominates $P \times \mathcal{Q}$. An application of Proposition 3.15 completes the proof. $\square$

At this point, the natural question becomes: *Does there exist a finitely dominated space Y that does not have finite homotopy type?* A version of this question was initially posed by Milnor in 1959 and answered affirmatively by Wall.

Theorem 3.18 (Wall's finiteness obstruction, [Wal65])**.** *For each finitely dominated space Y, there is a well-defined obstruction $\sigma(Y)$, lying in the reduced projective class group $\widetilde{K}_0(\mathbb{Z}[\pi_1(Y)])$, which vanishes if and only if Y has finite homotopy type.*

Moreover, all elements of $\widetilde{K}_0 \left(\mathbb{Z} \left[\pi_1 \left(Y \right) \right] \right)$ can be realized as finiteness obstructions of a finitely dominated CW complex.

A development of Wall's obstruction is interesting and entirely understandable, but outside the scope of these notes. The interested reader is referred to Wall's original paper or the exposition in [Fer].

Example 13. *Every compact ENR A is easily seen to be finitely dominated. Indeed, if $U \subseteq \mathbb{R}^n$ is a neighborhood of A and $r : U \to A$ a retraction, let $K \subseteq U$ be a polyhedral neighborhood of A, $d : K \to A$ the restriction, and u the inclusion.*
Although this is a nice example, it is made obsolete by a major result of West (see Proposition A.2), showing that every compact ANR has finite type.

3.5. **Inward tameness.** Modulo a slight change in terminology, we follow [ChSi76] by defining an ANR X to be *inward tame* if, for each neighborhood of infinity N there exists a smaller neighborhood of infinity N' so that, up to homotopy, the inclusion $N' \overset{j}{\hookrightarrow} N$ factors through a finite complex K. In other words, there exist maps $f : N' \to K$ and $g : K \to N$ such that $gf \simeq j$.

Exercise 3.19. *Show that if $X \overset{p}{\simeq} Y$ and X is inward tame, then Y is inward tame.*

For the remainder of this section, our goals are as follows:

a) obtain a more intrinsic and intuitive characterization of inward tameness, and

b) clarify the (apparent) relationship between inward tameness and finite dominations.

The following is our answer to Goal a).

Lemma 3.20. *An ANR X is inward tame if and only if, for every closed neighborhood of infinity N in X, there is a homotopy $K : N \times [0, 1] \to N$ with $K_0 = \mathrm{id}_N$ and $\overline{K_1 \left(N \right)}$ compact (a homotopy pulling N into a compact subset).*

Proof. For the forward implication, let N' be a closed neighborhood of infinity contained in int N so that $N' \hookrightarrow$ int N factors through a compact polyhedron K. Then there is a homotopy $H : N' \times [0, 1] \to$ int N with H_0 the inclusion and $\overline{H_1 \left(N' \right)} \subseteq g \left(K \right)$. Choose an open neighborhood U of N' with $\overline{U} \cap \mathrm{Bd}_X N = \varnothing$, then let $A = $ int $N - U$ and J be the identity homotopy on A. Since int N is an ANR, Borsuk's Homotopy Extension Property allows us to extend $H \cup J$ to a homotopy $K' : $ int $N \times [0, 1] \to$ int N with $K_0 = \mathrm{id}_{\mathrm{int}\, N}$. This in turn may be extended via the identity over $\mathrm{Bd}_X N$ to obtain a homotopy K that pulls N into a compact subset of itself.
We will return for the converse after addressing Goal b). $\square$

Recall that an ANR X is called sharp at infinity if it contains arbitrarily small closed ANR neighborhoods of infinity.

Lemma 3.21. *A space X that is sharp at infinity is inward tame if and only if each of its sharp neighborhoods of infinity is finitely dominated.*
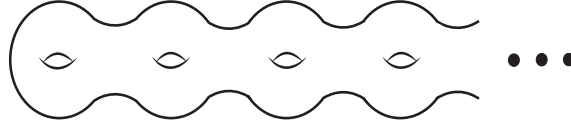
FIGURE 5. 1-ended infinite genus surface

Proof. Assume X is sharp at infinity and inward tame. By Lemma 3.20 each closed neighborhoods of infinity can be pulled into a compact subset, so by Proposition 3.17, those which are ANRs are finitely dominated. The converse is immediate by the definitions. □

Proof (completion of Lemma 3.20). Suppose that, for each closed neighborhood of infinity N in X, there is a homotopy pulling N into a compact subset. Then the same is true for $X \times \mathcal{Q}$. But $X \times \mathcal{Q}$ is sharp since it is a Hilbert cube manifold, so by Proposition 3.17, each ANR neighborhood of infinity in $X \times \mathcal{Q}$ is finitely dominated. By Lemma 3.21 and Exercise 3.19, X is inward tame. □

We tidy up by combining these Lemmas into a single Proposition, and adding some mild extensions. For convenience we restrict attention to spaces that are sharp at infinity.

Proposition 3.22. *For a space X that is sharp at infinity, the following are equivalent.*

(1) *X is inward tame,*
(2) *for every closed neighborhood of infinity N, there is a homotopy $H : N \times [0,1] \to N$ with $H_0 = \mathrm{id}_N$ and $\overline{H_1(N)}$ compact,*
(3) *there exist arbitrarily small closed neighborhood of infinity N, for which there is a homotopy $H : N \times [0,1] \to N$ with $H_0 = \mathrm{id}_N$ and $\overline{H_1(N)}$ compact,*
(4) *every sharp neighborhood of infinity is finitely dominated,*
(5) *there exist arbitrarily small sharp neighborhoods of infinity that are finitely dominated.*

Proof. The equivalence of (2) and (3) is by a homotopy extension argument like that found in Lemma 3.20. The equivalence of (4) and (5) is similar, but easier. □

Remark 3. The "inward" in inward tame is motivated by conditions (2) and (3) where the homotopies are viewed as pulling the end of X inward toward the center of X. Based on the definition and conditions (4) and (5), one may also think of inward tameness as "finitely dominated at infinity". We call X *absolutely inward tame* if it contains arbitrarily small closed ANR neighborhoods of infinity with finite type.

Example 14. *The infinite ladder $L_{[0,\infty)}$ is not inward tame, since its ANR neighborhoods of infinity have infinitely generated fundamental groups. Similarly, the infinite genus 1-ended orientable surface is not inward tame.*

Example 15. *Although the Whitehead manifold $\mathcal{W}^3$ itself has finite homotopy type, it is not inward tame, since the neighborhoods of infinity U_i discussed in Example 9*

do not have finitely generated fundamental groups (proof would require some work). The Davis manifolds, on the other hand, are absolutely inward tame. More on these observations in §5.3.

Exercise 3.23. *Justify the above assertion about the Davis manifolds.*

Example 16. *Every proper CAT(0) space X is absolutely inward tame. For inward tameness, let $N_{p,r}$ be the complement of an open ball $B(p;r)$ and use geodesics to strong deformation retract $N_{p,r}$ onto the metric sphere $S(p;r)$. If $S(p;r)$ is an ANR, then it (and thus $N_{p,r}$) have finite type by Proposition A.2. Since this is not known to be the case, more work is required. For each sharp neighborhood of infinity N, choose r so that $\overline{X - N} \subseteq B(p;r)$ and let $A = \overline{N - N_{p,r}}$. Then N strong deformation retracts onto A, which is a compact ANR.*

Before closing this section, we caution the reader that differing notions of "tameness" are scattered throughout the literature. [Sie65] called a 1-ended open manifold *tame* if it satisfies our definition for inward tame and also has stable pro-π_1. [ChSi76] reformulated tame to match our current-day definition of inward tame. Later still, [Qui88] and [HuRa96] put forth another version of "tame" in which homotopies push neighborhoods of infinity toward the end of the space—sometimes referring to that version as *forward tame* and the [ChSi76] version as *reverse tame*. In an effort to avoid confusion, this author introduced the term *inward tame,* while referring to the Quinn-Hughes-Ranicki version as *outward tame.*

Within the realm of 3-manifold topology, a *tame end* is often defined to a neighborhood of infinity $N \approx \partial N \times [0, \infty)$. Remarkably, by [Tuc74] combined with the 3-dimensional Poincaré conjecture, in the special case of 3-manifolds, this property, inward tame, and outward tame are all equivalent.

Despite the mildly confusing history, the concept of inward tameness (and its variants) is absolutely fundamental to the study of noncompact spaces. That will become clear as we proceed.

4. Algebraic invariants of the ends of a space: the precise definitions

In §2.2 we introduced the fundamental group at infinity rather informally. In this section we provide the details necessary to place that invariant on firm mathematical ground. In the process we begin to uncover subtleties that make this invariant even more interesting than one might initially expect.

As we progress, it will become apparent that the fundamental group at infinity (more precisely "pro-π_1") is just one of many "end invariants". By the end of the section, we will have introduced others, including pro-π_k and pro-H_k for all $k \geq 0$.

4.1. **An equivalence relation on the set of inverse sequences.** The *inverse limit* of an inverse sequence

$$ G_0 \xleftarrow{\mu_1} G_1 \xleftarrow{\mu_2} G_2 \xleftarrow{\mu_3} G_3 \xleftarrow{\mu_4} \cdots $$
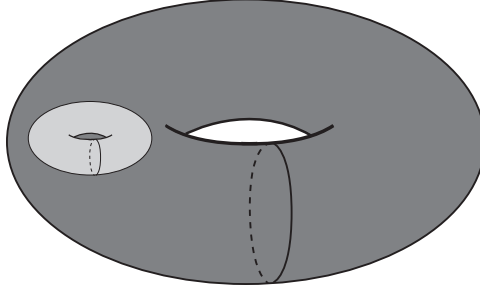
FIGURE 6. Exhausting $\mathbb{R}^3$ with solid tori.

of groups is defined by

$$\varprojlim \{G_i, \mu_i\} = \{(g_0, g_1, g_2, \cdots) \mid \mu_i(g_i) = g_{i-1} \text{ for all } i \geq 1\}.$$

Although useful at times, passing to an inverse limit often results in a loss of information. Instead, one usually opts to keep the entire sequence—or, more accurately, the essential elements of that sequence. To get a feeling for what is meant by "essential elements", let us look at some things that can go wrong.

In Example 5, we obtained the following representation of the fundamental group of infinity for $\mathbb{R}^3$.

(4.1) $$1 \leftarrow 1 \longleftarrow 1 \longleftarrow \cdots .$$

That was done by exhausting $\mathbb{R}^3$ with a sequence $\{i\mathbb{B}^3\}$ of closed i-balls and letting $U_i = \overline{\mathbb{R}^3 - i\mathbb{B}^3} \approx \mathbb{S}^2 \times [i, \infty)$. If instead, $\mathbb{R}^3$ is exhausted with a sequence $\{T_i\}$ of solid tori where each T_j lies in T_{j+1} as shown in Figure 6 and $V_i = \overline{\mathbb{R}^3 - T_i}$, the resulting representation of the fundamental group of infinity is

$$\mathbb{Z} \overset{0}{\leftarrow} \mathbb{Z} \overset{0}{\leftarrow} \mathbb{Z} \overset{0}{\leftarrow} \cdots .$$

By choosing more complicated exhausting sequences—for example, by higher genus knotted handlebodies—representations with even more complicated groups can be obtained. One can also arrange that the bonding homomorphisms are not always trivial. Yet each of these sequences purports to describe the same thing. Although it seems clear that (4.1) is the preferred representative for this example, in the case of an arbitrary 1-ended space, there may be no obvious best choice. The problem is resolved by placing an equivalence relation on the set of all inverse sequences of groups. Within an equivalence class, certain representatives may seem preferable to others, but each contains the essential information.

For an inverse sequence $\{G_i, \phi_i\}$, there is an obvious meaning for *subsequence*

$$G_{k_0} \overset{\phi_{k_0,k_1}}{\longleftarrow} G_{k_1} \overset{\phi_{k_1,k_2}}{\longleftarrow} G_{k_2} \overset{\phi_{k_2,k_3}}{\longleftarrow} \cdots$$

where the bonding homomorphisms $\phi_{k_i,k_{i+1}}$ are compositions of the ϕ_i. Declare inverse sequences $\{G_i, \phi_i\}$ and $\{H_i, \psi_i\}$ to be *pro-isomorphic* if they contain subsequences

that fit into a commuting "ladder diagram"

$$G_{i_0} \xleftarrow{\lambda_{i_0+1,i_1}} G_{i_1} \xleftarrow{\lambda_{i_1+1,i_2}} G_{i_2} \xleftarrow{\lambda_{i_2+1,i_3}} G_{i_3} \quad \cdots$$

$$H_{j_0} \xleftarrow{\mu_{j_0+1,j_1}} H_{j_1} \xleftarrow{\mu_{j_1+1,j_2}} H_{j_2} \xleftarrow{\mu_{j_2+1,j_3}} \cdots$$

More broadly, define *pro-isomorphism* to be the equivalence relation on the collection of all inverse sequences of groups generated by that rule[7].

It is immediate that an inverse sequence is pro-isomorphic to each of its subsequences; but sequences can appear very different and still be pro-isomorphic.

Exercise 4.1. *Convince yourself that the various inverse sequences mentioned above for describing the fundamental group at infinity of $\mathbb{R}^3$ are pro-isomorphic.*

Exercise 4.2. *Show that a pair of pro-isomorphic inverse sequences of groups have isomorphic inverse limits.* Hint. *Begin by observing a canonical isomorphism between the inverse limit of a sequence and that of any of its subsequences.*

The next exercise provides a counterexample to the converse of Exercise 4.2. It justifies our earlier assertion that passing to an inverse limit often results in loss of information.

Exercise 4.3. *Show that the inverse sequence $\mathbb{Z} \xleftarrow{\times 2} \mathbb{Z} \xleftarrow{\times 2} \mathbb{Z} \xleftarrow{\times 2} \cdots$ is not pro-isomorphic to the trivial inverse sequence $1 \leftarrow 1 \longleftarrow 1 \longleftarrow \cdots$, but both inverse limits are trivial.*

Exercise 4.4. *A more slick (if less intuitive) way to define pro-isomorphism is to declare it the equivalence relation generated by making sequences equivalent to their subsequences. Show that the two approaches are equivalent.*

Remark 4. With a little more work, we could define *morphisms* between inverse sequences of groups and arrive at a category *pro-Groups,* where the objects are inverse sequences of groups, in which two objects are pro-isomorphic if and only if they are isomorphic in that category.[8]

Similarly, for any category $\mathcal{C}$ one can build a category pro-$\mathcal{C}$ in which the objects are inverse sequences of objects and morphisms from $\mathcal{C}$ and for which the resulting relationship of pro-isomorphism is similar to the one defined above. All of this is interesting and useful, but more than we need here. For a comprehensive treatment of this topic, see [Geo08].

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[7]The prefix "pro" is derived from "projective". Some authors refer to inverse sequences and inverse limits as projective sequences and projective limits, respectively.

[8]We are not being entirely forthright here. In the literature, *pro-Groups* usually refers to a larger category consisting of "inverse systems" of groups indexed by arbitrary partially ordered sets. We have described a subcategory, *Tow-Groups*, made up of those objects indexed by the natural numbers—also known as "towers".

4.2. **Topological definitions and justification of the pro-isomorphism relation.** A quick look at the topological setting that leads to multiple inverse sequences representing the same fundamental group at infinity provides convincing justification for the definition of pro-isomorphic.

Let $U_0 \leftrightarrow U_1 \leftrightarrow U_2 \leftrightarrow \cdots$ and $V_0 \leftrightarrow V_1 \leftrightarrow V_2 \leftrightarrow \cdots$ be two cofinal sequences of connected neighborhoods of infinity for a 1-ended space X. By going out sufficiently far in the second sequence, one arrives at a V_{k_0} contained in U_0. Similarly, going out sufficiently far in the initial sequence produces a $U_{j_1} \subseteq V_{k_0}$. (For convenience, let $j_0 = 0$.) Alternating back and forth produces a ladder diagram of inclusions

(4.2)

$$V_{j_0} \longleftarrow V_{j_1} \longleftarrow V_{j_2} \longleftarrow \cdots$$
$$U_{k_0} \longleftarrow U_{k_1} \longleftarrow U_{k_2} \quad \cdots$$

Applying the fundamental group functor to that diagram (ignoring base points for the moment) results a diagram

(4.3)

$$\pi_1(V_{j_0}, q_{j_0}) \longleftarrow \pi_1(V_{j_1}, q_{j_1}) \longleftarrow \pi_1(V_{j_2}, q_{j_2}) \longleftarrow \cdots$$
$$\pi_1(U_{k_0}, p_{k_0}) \longleftarrow \pi_1(U_{k_1}, p_{k_1}) \longleftarrow \pi_1(U_{k_2}, p_{k_2}) \quad \cdots$$

showing that $\pi_1(U_0) \xleftarrow{\lambda_1} \pi_1(U_1) \xleftarrow{\lambda_2} \pi_1(U_2) \xleftarrow{\lambda_3} \cdots$ and $\pi_1(V_0) \xleftarrow{\mu_1} \pi_1(V_1) \xleftarrow{\mu_2} \pi_1(V_2) \xleftarrow{\mu_3} \cdots$ are pro-isomorphic.

A close look at base points and base rays is still to come, but recognizing their necessity, we make the following precise definition. For a pair (X, r) where r is a proper ray in X, let pro-$\pi_1(\varepsilon(X), r)$ denote the pro-isomorphism class of inverse sequences of groups which contains representatives of the form (2.1), where $\{U_i\}_{i=0}^{\infty}$ is a cofinal sequence of neighborhoods of infinity, and r has been modified (in the manner described in Exercise 3.12) so that $r([i, \infty)) \subseteq U_i$ for each $i \geq 0$. From now on, when we refer to the *fundamental group at infinity (based at r)* of a space X, we mean pro-$\pi_1(\varepsilon(X), r)$.

With the help of Exercise (4.2), we also define the *Čech fundamental group of the end of X (determined r)*, to be the inverse limit of pro-$\pi_1(\varepsilon(X), r)$. It is denoted by $\check{\pi}_1(\varepsilon(X), r)$.

Exercise 4.5. *Fill in the details related to base points and base rays needed for the existence of diagram (4.3).*

Remark 5. Now that pro-$\pi_1(\varepsilon(X), r)$ is well-defined and (hopefully) well-understood for 1-ended X, it is time to point out that everything done thus far works for multi-ended X. In those situations, the role of r is more pronounced. In the process of selecting base points for a sequence of neighborhoods of infinity $\{U_i\}$, r determines the component of each U_i that contributes its fundamental group to pro-$\pi_1(\varepsilon(X), r)$.

So, if r and s point to different ends of X, pro-$\pi_1(\varepsilon(X), r)$ and pro-$\pi_1(\varepsilon(X), s)$ reflect information about entirely different portions of X. This observation is just the beginning; a thorough examination of the role of base rays is begun in §4.6.

4.3. **Other algebraic invariants of the end of a space.** By now it has likely occurred to the readers that π_1 is not the only functor that can be applied to an inverse sequence of neighborhoods of infinity. For any $k \geq 1$ and proper ray r, define pro-$\pi_k(\varepsilon(X), r)$ in the analogous manner. By taking inverse limits we get the *Čech homotopy groups* $\check{\pi}_k(\varepsilon(X), r)$ of the end of X determined by r. Similarly, we may define pro-$\pi_0(\varepsilon(X), r)$ and $\check{\pi}_0(\varepsilon(X), r)$; the latter is just a set, and the former an equivalence class of inverse sequences of sets.

By applying the homology functor we obtain pro-$H_k(\varepsilon(X); R)$ and $\check{H}_k(\varepsilon(X); R)$ for each non-negative integer k and arbitrary coefficient ring R, the latter being called the *Čech homology of the end of X*. In this context, no base ray is needed!

If instead we apply the cohomology functor, a significant change occurs. The contravariant nature of H^k produces *direct sequences*

$$H^k(U_0; R) \xrightarrow{\lambda_1} H^k(U_1; R) \xrightarrow{\lambda_2} H^k(U_2; R) \xrightarrow{\lambda_3} \cdots$$

of cohomology groups. An algebraic treatment of such sequences, paralleling §4.1, and a standard definition of *direct limit*, allow us to define ind-$H^*(\varepsilon(X); R)$ and $\check{H}^*(\varepsilon(X); R)$.

Exercise 4.6. *Show that for ANRs there is a one-to-one correspondence between* $\mathcal{E}\mathrm{nds}(X)$ *and* $\check{\pi}_0(\varepsilon(X), r)$.

4.4. **End invariants and the proper homotopy category.** In Remark 2, we commented on the importance of proper maps and proper homotopy equivalences in the study of noncompact spaces. We are now ready to back up that assertion. The following Proposition could be made even stronger with a discussion of morphisms in the category of pro-Groups, but for our purposes, it will suffice.

Proposition 4.7. *Let $f : X \to Y$ be a proper homotopy equivalence and r a proper ray in X. Then*

    (1) pro-$H_k(X; R)$ *is pro-isomorphic to* pro-$H_k(Y; R)$ *for all k and every coefficient ring R,*
    (2) pro-$\pi_0(X)$ *is pro-isomorphic to* pro-$\pi_0(Y)$ *as sets, and*
    (3) pro-$\pi_k(X, r)$ *is pro-isomorphic to* pro-$\pi_k(Y, f \circ r)$ *for all $k \geq 1$.*

Corollary 4.8. *A proper homotopy equivalence $f : X \to Y$ induces isomorphisms between $\check{H}_k(X; R)$ and $\check{H}_k(Y; R)$ for all k and every coefficient ring R. It induces a bijection between $\check{\pi}_0(X)$ and $\check{\pi}_0(Y)$ and isomorphisms between $\check{\pi}_k(X, r)$ and $\check{\pi}_k(Y, f \circ r)$ for all $k \geq 1$.*

Sketch of the proof of Proposition 4.7. Let $g : Y \to X$ be a proper inverse for f and let H and K be proper homotopies between $g \circ f$ and id_X and $f \circ g$ and id_Y, respectively. By using the properness of H and K and a back-and-forth strategy similar to
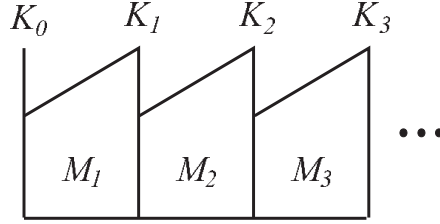
FIGURE 7. The infinite mapping telescope $\mathrm{Tel}\left(\{K_i, f_i\}\right)$

the one employed in obtaining diagram (4.2), we obtain systems of neighborhoods of infinity $\{U_i\}$ in X and $\{V_i\}$ in Y that fit into a ladder diagram

(4.4)

$$V_{j_0} \longleftarrow V_{j_1} \longleftarrow V_{j_2} \longleftarrow \cdots$$
$$U_{k_0} \longleftarrow U_{k_1} \longleftarrow U_{k_2} \quad \cdots$$

Unlike the earlier case, the up and down arrows are not inclusions, but rather restrictions of f and g. Furthermore, the diagram does not commute on the nose; instead, it commutes up to homotopy. But that is enough to obtain a commuting ladder diagram of homology groups, thus verifying 1). The same is true for (2), but on the level of sets. Assertion (3) is similar, but a little additional care must be taken to account for the base rays. $\qquad\square$

4.5. **Inverse mapping telescopes and a topological realization theorem.** It is natural to ask which inverse sequences (more precisely, pro-isomorphism classes) can occur as pro-$\pi_1\left(\varepsilon(X), r\right)$ for a space X. Here we show that, even if restricted to very nice spaces, the answer is "nearly all of them". Later we will see that, in certain important contexts the answer becomes much different. But for now we create a simple machine for producing wide range of examples.

Let

(4.5) $$(K_0, p_0) \xleftarrow{\ f_1\ } (K_1, p_1) \xleftarrow{\ f_2\ } (K_2, p_2) \xleftarrow{\ f_3\ } \cdots$$

be an inverse sequence of pointed finite CW complexes and cellular maps. For each $i \geq 1$, let M_i be a copy of the *mapping cylinder* of f_i; more specifically

$$M_i = (K_i \times [i-1, i]) \sqcup (K_{i-1} \times \{i-1\})/\sim_i$$

where $\sim_i$ is the equivalence relation generated by the rule: $(k, i-1) \sim_i (f_i(k), i-1)$ for each $k \in K_i$. Then M_i contains a canonical copy $K_{i-1} \times \{i-1\}$ of K_{i-1} and a canonical copy $K_i \times \{i\}$ of K_i; and $M_{i-1} \cap M_i = K_{i-1} \times \{i-1\}$. The infinite union $\mathrm{Tel}\left(\{K_i, f_i\}\right) = \cup_{i=1}^{\infty} M_i$, with the obvious topology is called the *mapping telescope* of (4.5). See Figure 7.

For each $x \in K_i$, the (embedded) copy of the interval $\{x\} \times [i-1, i]$ in M_i is called a *mapping cylinder line*. The following observations are straightforward.

- Tel $(\{K_i, f_i\})$ may be viewed as the union of infinite and dead end "telescope rays", each of which begins in $K_0 \times \{0\}$ and intersect a given M_i in a mapping cylinder line or not at all. The dead end rays and empty intersections occur only when a point $k \in K_j$ is not in the image of f_{j+1}; whereas, the infinite telescope rays are proper and in one-to-one correspondence with $\varprojlim \{K_i, f_i\}$,
- by choosing a canonical set of strong deformation retractions of the above rays to their initial points, one obtains a strong deformation retraction of Tel $(\{K_i, f_i\})$ to $K_0 \times \{0\}$.
- letting $U_k = \cup_{i=k+1}^{\infty} M_i$ provides a cofinal sequence of neighborhoods of infinity. By a small variation on the previous observation each $K_i \times \{i\} \hookrightarrow U_i$ is a homotopy equivalence. (So Tel $(\{K_i, f_i\})$ is absolutely inward tame.)
- letting r be the proper ray consisting of the cylinder lines connecting each p_i to p_{i-1}, we obtain a representation of pro-$\pi_1 (\varepsilon(X), r)$ which is pro-isomorphic to the sequence

$$\pi_1 (K_0, p_0) \xleftarrow{f_{1\#}} \pi_1 (K_1, p_1) \xleftarrow{f_{2\#}} \pi_1 (K_2, p_2) \xleftarrow{f_{3\#}} \cdots.$$

Proposition 4.9. *For every inverse sequence $G_0 \xleftarrow{\mu_1} G_1 \xleftarrow{\mu_2} G_2 \xleftarrow{\mu_3} \cdots$ of finitely presented groups, there exists a 1-ended, absolutely inward tame, locally finite CW complex X and a proper ray r such that pro-$\pi_1 (\varepsilon(X), r)$ is represented by that sequence. If desired, X can be chosen to be contractible.*

Proof. For each i, let K_i be a presentation 2-complex for G_i and let $f_i : K_i \to K_{i-1}$ be a cellular map that induces μ_i. Then let $X = \mathrm{Tel}\,(\{K_i, f_i\})$.

In order to make X contractible, one simply adds a trivial space $K_{-1} = \{p_{-1}\}$ to the left end of the sequence of complexes. $\square$

Example 17. *An easy application of Proposition 4.9 produces a pro-$\pi_1 (X, r)$ equal to the inverse sequence $\mathbb{Z} \xleftarrow{\times 2} \mathbb{Z} \xleftarrow{\times 2} \mathbb{Z} \xleftarrow{\times 2} \cdots$ discussed in Exercise (4.3). For each i, let $\mathbb{S}_i^1$ be a copy of the unit circle and $f_i : \mathbb{S}_i^1 \to \mathbb{S}_{i-1}^1$ the standard degree 2 map. Then $X = \mathrm{Tel}\,(\{\mathbb{S}_i^1, f_i\})$ is 1-ended and has the desired fundamental group at infinity.*

Proposition 4.10. *For every inverse sequence $G_0 \xleftarrow{\mu_1} G_1 \xleftarrow{\mu_2} G_2 \xleftarrow{\mu_3} \cdots$ of finitely presented groups and $n \geq 6$, there exists a 1-ended open n-manifold M^n such that pro-$\pi_1 (M^n, r)$ is represented by that sequence. If a (noncompact) boundary is permitted, and $n \geq 7$, then M^n can be chosen to be contractible.*

Proof. Let $X = \mathrm{Tel}\,(\{K_i, f_i\})$ as constructed in the previous Proposition. With some extra care, arrange for X to be a simplicial 3-complex, and choose a proper PL embedding into $\mathbb{R}^{n+1}$. Let N^{n+1} be a regular neighborhood of that embedding. It is easy to see that pro-$\pi_1(N^{n+1}, r)$ is identical to pro-$\pi_1 (X, r)$, so if boundary is permitted, we are finished. If not, let $M^n = \partial N^{n+1}$. By general position, the base ray r may be slipped off X and then isotoped to a ray r' in M^n. Also by general position, loops and disks in N^{n+1} may be slipped off X and then pushed into M^n. In doing so, one sees that pro-$\pi_1 (M^n, r')$ is pro-isomorphic to pro-$\pi_1 (N^{n+1}, r)$. $\square$

In the study of compact manifolds, results like Poincaré duality place significant restrictions on the topology of closed manifolds. A similar phenomenon occurs in the study of noncompact manifolds. In that setting, it is the open manifolds (and to a similar extent, manifolds with compact boundary) that are the more rigidly restricted. If an open manifold is required to satisfy additional niceness conditions, such as contractibility, finite homotopy type, or inward tameness, even more rigidity comes into play. This is at the heart of the study of noncompact manifolds, where a goal is to obtain strong conclusions about the structure of a manifold from modest hypotheses.

Exercise 4.11. *Show that an inward tame manifold M^n with compact boundary cannot have infinitely many ends. (*Hint. *Homology with $\mathbb{Z}_2$-coefficients simplifies the algebra and eliminates issues related to orientability.) Show that this result fails if omit the tameness hypothesis or if M^n is permitted to have noncompact boundary.*

Exercise 4.12. *Show that the inverse sequence realized in Example 17 cannot occur as pro-$\pi_1 (M^n, r)$ for a contractible open manifold.* **Hint.** *A look ahead to §5.1 may be helpful.*

The trick used in the proof of Proposition 4.9 for obtaining a *contractible* mapping telescope with the same end behavior as one that is homotopically nontrivial is often useful. Given an inverse sequence $\{K_i\}$ of finite CW complexes, the *augmented inverse sequence* $\{K_i, f_i\}^\bullet$ is obtained by inserting a singleton space at the beginning of $\{K_i, f_i\}$; the corresponding *contractible mapping telescope* $\mathrm{CTel}\left(\{K_i, f_i\}\right)$ is contractible, but identical to $\mathrm{Tel}\left(\{K_i, f_i\}\right)$ at infinity.

4.6. **On the role of the base ray.** We now begin the detailed discussion of the role of base rays in the fundamental group at infinity—a topic more subtle and more interesting than one might expect.

As hinted at earlier small changes in base ray, such as reparameterization or deletion of an initial segment, do not alter pro-$\pi_1 \left(\varepsilon (X), r\right)$; this follows from a more general result to be presented shortly. On the other hand, large changes can obviously have an impact. For example, if X is multi-ended and r and s point to different ends, then pro-$\pi_1 \left(\varepsilon (X), r\right)$ and pro-$\pi_1 \left(\varepsilon (X), s\right)$ provide information about different portions of X —much as the traditional fundamental group of a non-path-connected space provides different information when the base point is moved from one component to another. When r and s point to the same end of X, it is reasonable to expect pro-$\pi_1 \left(\varepsilon (X), r\right)$ and pro-$\pi_1 \left(\varepsilon (X), s\right)$ to be pro-isomorphic—but this is not the case either! At the heart of the matter is the difference between the set of ends $\mathcal{E}(X)$ and the set of strong ends $\mathcal{SE}(X)$. The following requires some effort, but the proof is completely elementary.

Proposition 4.13. *If proper rays r and s in X are strongly equivalent, ie, properly homotopic, then pro-$\pi_1 \left(\varepsilon (X), r\right)$ and pro-$\pi_1 \left(\varepsilon (X), s\right)$ are pro-isomorphic.*

Corollary 4.14. *If X is strongly connected at infinity, ie, $|\mathcal{SE}(X)| = 1$, then pro-$\pi_1 \left(\varepsilon (X)\right)$ is a well-defined invariant of X.*

Exercise 4.15. *Prove Proposition 4.13.*

Remark 6. There are useful analogies between the role played by base points in the fundamental group and that played by base rays in the fundamental group at infinity:

- The fundamental group is best viewed as a functor from the category of *pointed spaces,* ie, pairs (Y, p), where $p \in Y$, to the category of groups. The fundamental group of infinity is a functor from the proper category of pairs (X, r), where r is a proper ray in X, to the category pro-Groups.
- If there is a path α in Y from p to q in Y, there is a corresponding isomorphism $\widehat{\alpha} : \pi_1(Y, p) \to \pi_1(Y, q)$. If there is a proper homotopy in X between proper rays r and s, then there is a corresponding pro-isomorphism between pro-$\pi_1(\varepsilon(X), r)$ and pro-$\pi_1(\varepsilon(X), s)$.
- Even for connected Y there may be no relationship between $\pi_1(Y, p)$ and $\pi_1(Y, q)$ when there is no path connecting p to q. Similarly, for a 1-ended space X, pro-$\pi_1(\varepsilon(X), r)$ and pro-$\pi_1(\varepsilon(X), s)$ may be very different if there is no proper homotopy from r to s.

We wish to describe a 1-ended Y with proper rays r and s for which pro-$\pi_1(\varepsilon(X), r)$ and pro-$\pi_1(\varepsilon(X), s)$ are not pro-isomorphic. We begin with an intermediate space.

Example 18 (Another space with $\mathcal{SE}(X) \neq \mathcal{E}(X)$). *Let $X = \mathrm{CTel}(\{\mathbb{S}_i^1, f_i\})$ where each $\mathbb{S}_i^1$ is a copy of the unit circle and $f_i : \mathbb{S}_i^1 \to \mathbb{S}_{i-1}^1$ is the standard degree 2 map (see Example 17). If p_i is the canonical base point for the ith copy of $\mathbb{S}^1$ and $f_i(p_i) = p_{i-1}$ for all i, we may construct a "straight" proper ray r by concatenating the mapping cylinder lines α_i connecting p_i and p_{i-1}. Construct a second proper ray s by splicing between each α_i and α_{i+1} a loop β_i that goes once in the positive direction around $\mathbb{S}_i^1$; in other words, $s = \alpha_0 \cdot \beta_0 \cdot \alpha_1 \cdot \beta_1 \cdot \alpha_2 \cdots$. With some effort, it can be shown that r and s are not properly homotopic. That observation is also a corollary of the next example.*

Example 19. *For each i, let K_i be a wedge of two circles and let $g_i : K_i \to K_{i-1}$ send one of those circles onto itself by the identity and the other onto itself via the standard degree 2 map. Let $Y = \mathrm{CTel}(\{K_i, g_i\})$. This space may be viewed as the union of X from Example 18 and an infinite cylinder $\mathbb{S}^1 \times [0, \infty)$, coned off at the left end, with the union identifying the ray r with a standard ray in the product. By viewing X as a subset of Y, view r and s as proper rays in Y.*

Choose neighborhoods of infinity U_i as described in § 4.5. Each has fundamental group that is free of rank 2. If we let F_i be the free group of rank 2 with formal generators a^{2i} and b then, pro-$\pi_1(\varepsilon(Y), r)$ may be represented by

$$\langle a, b \rangle \leftarrow \langle a^2, b \rangle \leftarrow \langle a^4, b \rangle \leftarrow \cdots .$$

Similarly pro-$\pi_1(\varepsilon(Y), s)$ may be represented by

$$\langle a, b \rangle \xleftarrow{\lambda_1} \langle a^2, b \rangle \xleftarrow{\lambda_2} \langle a^4, b \rangle \xleftarrow{\lambda_3} \cdots .$$

where $\lambda_i\left(a^{2i}\right) = a^{2i}$ and $\lambda_i\left(b\right) = a^{2i}ba^{-2i}$. Taking inverse limits, produces $\check{\pi}_1\left(\varepsilon\left(Y\right), r\right)$ $= \langle b \rangle \cong \mathbb{Z}$ and $\check{\pi}_1\left(\varepsilon\left(Y\right), s\right) = 1$. Hence pro-$\pi_1\left(\varepsilon\left(Y\right), r\right)$ and pro-$\pi_1\left(\varepsilon\left(Y\right), s\right)$ are not pro-isomorphic.

Exercise 4.16. *Verify the assertions made in each of the two previous Examples.*

The fact that a 1-ended space can have multiple fundamental groups at infinity might lead one to doubt the value of that invariant. Over the next several sections we provide evidence to counter that impression. For example, will investigate some properties of pro-π_1 that persist under change of base ray. Furthermore, we will see that in some of the most important situations, there is (verifiably in some cases and conjecturally in others) just one proper homotopy class of base ray—causing the ambiguity to vanish. As an example, the following important question is open.

Conjecture 1 (The Manifold Semistability Conjecture–version 1)**.** *The universal cover of a closed aspherical manifold of dimension greater than 1 is always strongly connected at infinity?*

We stated the above problem as a conjecture because it is a special case of the following better-known conjecture. For now the reader can guess the necessary definitions. The meaning and will be fully explained in §6. The naming of these conjectures will become explained over the next couple pages.

Conjecture 2 (The Semistability Conjecture-version 1)**.** *Every finitely presented 1-ended group is strongly connected at infinity.*

4.7. **Flavors of inverse sequences of groups.** When dealing with pro-isomorphism classes of inverse sequences of groups, general properties are often more significant than the sequences themselves. In this section we discuss several such properties.

Let $G_0 \xleftarrow{\mu_1} G_1 \xleftarrow{\mu_2} G_2 \xleftarrow{\mu_3} G_2 \xleftarrow{\mu_4} \cdots$ be an inverse sequence of groups. We say that $\{G_i, \mu_i\}$ is

- *pro-trivial* if it is pro-isomorphic to the trivial inverse sequence $1 \leftarrow 1 \leftarrow 1 \leftarrow 1 \leftarrow \cdots$,
- *stable* if it is pro-isomorphic to an inverse sequence $\{H_i, \lambda_i\}$ where each λ_i is an isomorphism, or equivalently, a constant inverse sequence $\{H, \mathrm{id}_H\}$,
- *semistable* (or *Mittag-Leffler,* or *pro-epimorphic*) if it is pro-isomorphic to an $\{H_i, \lambda_i\}$, where each λ_i is an epimorphism, and
- *pro-monomorphic* if it is pro-isomorphic to an $\{H_i, \lambda_i\}$, where each λ_i is a monomorphism.

The following easy exercise will help the reader develop intuition for the above definitions, and for the notion of pro-isomorphism itself.

Exercise 4.17. *Show that an inverse sequence of non-injective epimorphisms cannot be pro-monomorphic, and that an inverse sequence of non-surjective monomorphisms cannot be semistable.*

Exercise 4.18. *Show that if $\{G_i, \mu_i\}$ is stable and thus pro-isomorphic to some $\{H, \mathrm{id}_H\}$, then H is well-defined up to isomorphism. In that case $H \cong \varprojlim \{G_i, \mu_i\}$.*

A troubling aspect of the above definitions is that the concepts appear to be extrinsic, requiring a second unseen sequence, rather being intrinsic to a given sequence. A standard result corrects that misperception.

Proposition 4.19. *An inverse sequence of groups $\{G_i, \lambda_i\}$ is stable if and only if it contains a subsequence for which "passing to images" results in an inverse sequence of isomorphisms, in other words; we may obtain a diagram of the following form, where all unlabeled homomorphisms are obtained by restriction or inclusion.*

(4.6)

$$
\begin{array}{ccccccc}
G_{i_0} & \xleftarrow{\lambda_{i_0+1,i_1}} & G_{i_1} & \xleftarrow{\lambda_{i_1+1,i_2}} & G_{i_2} & \xleftarrow{\lambda_{i_2+1,i_3}} & G_{i_3} \quad \cdots \\
\end{array}
$$

$$
\mathrm{Im}\,(\lambda_{i_0+1,i_1}) \xleftarrow{\ \cong\ } \mathrm{Im}\,(\lambda_{i_1+1,i_2}) \xleftarrow{\ \cong\ } \mathrm{Im}\,(\lambda_{i_2+1,i_3}) \xleftarrow{\ \cong\ } \cdots
$$

Analogous statements are true for the pro-epimorphic and pro-monomorphic sequences; in those cases we require maps in the bottom row of 4.6 to be epimorphisms, and monomorphisms, respectively.

Proof of the above is an elementary exercise, as is the following:

Proposition 4.20. *An inverse sequence is stable if and only if it is both pro-epimorphic and pro-monomorphic.*

Exercise 4.21. *Prove the previous two propositions.*

4.8. **Some topological interpretations of the above definitions.** It is common practice to characterize simply connected spaces topologically (without mentioning the word 'group'), as path-connected spaces in which every loop contracts to a point. In that spirit, we provide topological characterizations of spaces whose fundamental groups at infinity possess some of the algebraic properties discussed in the previous section.

Proposition 4.22. *For a 1-ended space X and a proper ray r, pro-$\pi_1\left(\varepsilon\left(X\right), r\right)$ is*
   (1) *pro-trivial if and only if: for any compact $C \subseteq X$, there exists a larger compact set D such that every loop in $X - D$ contracts in $X - C$,*
   (2) *semistable if and only if: for any compact $C \subseteq X$, there exists a larger compact set D such that, for every still larger compact E, each pointed loop α in $X - D$ based on r can be homotoped into $X - E$ via a homotopy into $X - C$ that slides the base point along r, and*
   (3) *pro-monomorphic if and only if there exists a compact $C \subseteq X$ such that, for every compact set D containing C, there exists a compact E such that every loop in $X - E$ that contracts in $X - C$ contracts in $X - D$.*

Proof. This is a straightforward exercise made easier by applying Proposition 4.19.
$\square$

Note that the topological condition in part (1) of Proposition 4.22 makes no mention of a base ray. So (for 1-ended spaces) the property of having pro-trivial fundamental group at infinity is independent of base ray; such spaces are called *simply connected at infinity*. Similarly, the topological condition in (3) is independent of base ray; 1-ended spaces with that property are called *pro-monomorphic at infinity* (or simply *pro-monomorphic*). And despite the (unavoidable) presence of a base ray in the topological portion of (2), there does exist an elegant and useful characterization of spaces with semistable pro-π_1.

Proposition 4.23. *A 1-ended space X is strongly connected at infinity if and only if there exists a proper ray r for which* pro-$\pi_1(X, r)$ *is semistable.*

Sketch of proof. First we outline a proof of the reverse implication. Let r be as in the hypothesis and let s be another proper ray. By 1-endedness, there is a proper map h of the infinite ladder $L_{[0,\infty)} = ([0,\infty) \times \{0,1\}) \cup (\mathbb{N} \times [0,1])$ into X, with $h|_{[0,\infty) \times 0} = r$ and $h|_{[0,\infty) \times 1} = s$. For convenience, choose an exhaustion of X by compacta $\varnothing = C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots$ with the property that the subladder $L_{[i,\infty)}$ is sent into $U_i = \overline{X - C_i}$ for each $i \geq 1$. As a simplifying hypothesis, assume that all bonding homomorphisms in the corresponding inverse sequence

$$(4.7) \qquad \pi_1(X, p_0) \xleftarrow{\lambda_1} \pi_1(X - C_1, p_1) \xleftarrow{\lambda_2} \pi_1(X - C_2, p_2) \xleftarrow{\lambda_3} \cdots$$

are surjective. (For a complete proof, one should instead apply Proposition 4.22 inductively.)

We would like to extend h to a proper map of $[0,\infty) \times [0,1]$ into X. To that end, let $\square_i$ be the loop in X corresponding to $r_{i+1} \cup e_{i+1} \cup s_{i+1}^{-1} \cup e_i^{-1}$ in $L_{[0,\infty)}$. (Here $r_{i+1} = r|_{[i,i+1]}$ and $s_{i+1} = s|_{[i,i+1]}$; $e_j = h|_{j \times [0,1]}$, the jth "rung" of the ladder.)

If each $\square_i$ contracts in X we can use those contractions to extend h to $[0,\infty) \times [0,1]$; if each $\square_i$ contracts in $X - C_i$ the resulting extension is proper (as required). The idea of the proof is to arrange those conditions. Begin inductively with $\square_0$. If this loop does not contract in X, we make it so by rechoosing e_1 as follows: choose a loop α_1 based at p_1 so that $r_1 \cdot \alpha_1 \cdot r_1^{-1}$ is equal to $\square_0$ in $\pi_1(X, p_0)$. Replace e_1 with the rung $\hat{e}_1 = \alpha_1^{-1} \cdot e_1$. The newly modified $\square_0$ contracts in X, as desired. Now move to the correspondingly modified $\square_1$ viewed as an element of $\pi_1(X - C_1, p_1)$. If it is nontrivial, choose a loop α_2 in $X - C_2$ based at p_2 such that $\lambda_2(\alpha_2) = r_2 \cdot \alpha_2 \cdot r_2^{-1} = \square_1$. Replacing e_2 with $\hat{e}_2 = \alpha_2^{-1} \cdot e_2$ results in a further modified $\square_1$ that contracts in $X - C_1$. Continue this process inductively to obtain a proper homotopy $H : [0,\infty) \times [0,1] \to X$ between r and s.

For the reverse implication, assume that (4.7) is not semistable. One creates a proper ray s not properly homotopic to r by affixing to each vertex p_i of r a loop in β_i in $X - C_i$ that that does not lie in the image of $\pi_1(X - C_{i+1}, p_{i+1})$. More specifically

$$s = r_1 \cdot \alpha_1 \cdot r_2 \cdot \alpha_2 \cdot r_3 \cdot \alpha_3 \cdots .$$

$\square$

As a result of Proposition 4.23, a 1-ended space X may be called *semistable at infinity* [respectively, *stable at infinity*] if pro-$\pi_1(X, r)$ is semistable [respectively,

stable] for some (and hence any) proper ray r. Alternatively, a 1-ended space is sometimes *defined* to be semistable at infinity (or just *semistable*) if all proper rays in X are properly homotopic. In those cases we often drop the base ray and refer to the homotopy end invariants simply as pro-$\pi_1(\varepsilon(X))$ and $\check{\pi}_1(\varepsilon(X))$.

Multi-ended spaces are sometimes called semistable if, whenever two proper rays determine the same end, they are properly homotopic; or equivalently, when $\Phi : \mathcal{SE}(X) \to \mathcal{E}(X)$ is bijective.

Remark 7. By using the sketched proof of Proposition 4.23 as a guide, it is not hard to see why a 1-ended space X that is not semistable will necessarily have *uncountable* $\mathcal{SE}(X)$. A method for placing $\mathcal{SE}(X)$ into an algebraic context involves the *derived limit* or '$\lim^1$ *functor*'. More generally, $\lim^1\{G_i, \mu_i\}$ is an algebraic construct that helps to recover the information lost when one passes from an inverse sequence to its inverse limit. See [Geo08, §11.3].

5. Applications of end invariants to manifold topology

Although a formal study of pro-homotopy and pro-homology of the ends of non-compact space is not a standard part of the education of most manifold topologists, there are numerous important results and open questions best understood in that context. In this section we discuss several of those, beginning with classical results and moving toward recent work and still-open questions.

5.1. **Another look at contractible open manifolds.** We now return to the study of contractible open manifolds begun in §2. We will tie up some loose ends from those earlier discussions mostly—most of which focused on specific examples. We also present some general results whose hypotheses involve nothing more that the fundamental group at infinity.

Theorem 5.1 (Whitehead's Exotic Open 3-manifold). *There exists a contractible open 3-manifold not homeomorphic to $\mathbb{R}^3$.*

Proof. We wish to nail down a proof that the Whitehead contractible 3-manifold $\mathcal{W}^3$ described in §2.1 is not homeomorphic to $\mathbb{R}^3$. We do that by showing $\mathcal{W}^3$ is not simply connected at infinity. Using the representation of pro-$\pi_1(\varepsilon(\mathcal{W}^3), r)$ obtained in §2.2 and applying the rigorous development from §4, we can accomplish that task with an application of Exercise 4.17. $\square$

Theorem 5.2. *The open Newman contractible n-manifolds are not homeomorphic to $\mathbb{R}^n$. More generally, any compact contractible n-manifold with non-simply connected boundary has interior that is not homeomorphic to $\mathbb{R}^n$.*

Proof. Combine our observations from Example 6 with Exercise 4.18—or simply observe that the topological characterization of simply connected at infinity fails. $\square$

The next result established simple connectivity at infinity as the definitive property in determining whether a contractible open manifold is exotic. The proof for PL manifolds of dimension ≥ 5 is clean, elegant, and highly recommended—but outside

the scope of these notes. Not surprisingly, in dimensions 3 and 4 it requires the Fields medal winning work of Perelman and Freedman, respectively.

Theorem 5.3 (Stallings' Characterization of $\mathbb{R}^n$, [Sta62]). *A contractible open n-manifold ($n \geq 3$) is homeomorphic to $\mathbb{R}^n$ if and only if it is simply connected at infinity.*

Exercise 5.4. *Prove the following corollary to Theorem 5.3: If W^n is a contractible open manifold, then $W^n \times \mathbb{R} \approx \mathbb{R}^{n+1}$.*

The next application of the fundamental group at infinity returns us to another prior discussion.

Theorem 5.5 (Davis' Exotic Universal Covering Spaces). *For $n \geq 4$, there exist closed aspherical n-manifolds whose universal covers are not homeomorphic to $\mathbb{R}^n$.*

Proof. Here we provide only the punch-line to this major theorem. As noted in §2.1 Davis' construction produces closed aspherical n-manifolds M^n with universal covers homeomorphic to the infinite open sums described in Example 3 and Theorem 2.3. As observed in Example 7, pro-$\pi_1 \left(\widetilde{M^n}, r \right)$ may be represented by

(5.1) $$G \leftarrow G * G \leftarrow G * G * G \leftarrow G * G * G * G \leftarrow \cdots,$$

a sequence that is semistable but not pro-monomorphic. Another application of Exercise 4.17 verifies that $\widetilde{M^n}$ is not simply connected at infinity. $\square$

After Davis showed that aspherical manifolds need not be covered by $\mathbb{R}^n$, many questions remained. With the 3-dimensional version unresolved (at the time), it was asked whether the Whitehead manifold could cover a closed 3-manifold. In higher dimensions, people wondered whether a Newman contractible open manifold (or the interior of another compact contractible manifold) could cover a closed manifold. Myers [Mye88] resolved the first question in the negative, before Wright proved a remarkably general result in which the fundamental group at infinity plays a central role.

Theorem 5.6 (Wright's Covering space Theorem). *Let M^n be a contractible open n-manifold with pro-monomorphic fundamental group at infinity. If M^n admits a nontrivial action by covering transformations, then $M^n \approx \mathbb{R}^n$.*

Corollary 5.7. *Neither the Whitehead manifold nor the interior of any compact contractible manifold with non-simply connected boundary can cover a manifold non-trivially.*

Wright's theorem refocuses attention on a question mentioned earlier.

Conjecture 3 (The Manifold Semistability Conjecture). *Must the universal cover of every closed aspherical manifold have semistable fundamental group at infinity?*

More generally we can ask:

Vague Question: *Must all universal covers of aspherical manifolds be similar to the Davis examples?*

FIGURE 8. $\mathbb{R}^n$ connect-summed with infinitely many $\mathbb{S}^2 \times \mathbb{S}^{n-2}$

In discussions still to come, we will make this vague question more precise. But, before moving on, we note that in 1991 Davis and Januszkiewicz [DaJa91] invented a new strategy for creating closed aspherical manifolds with exotic universal covers. Although that strategy is very different from Davis' original approach, the resulting exotic covers are remarkably similar. For example, their fundamental groups at infinity are precisely of the form (5.5).

Exercise 5.8. *Theorem 5.3 suggests that the essence of a contractible open manifold is contained in its fundamental group at infinity. Show that every contractible open n-manifold W^n has the same* homology *at infinity as $\mathbb{R}^n$. In particular, show that pro-$H_i(W^n; \mathbb{Z})$ is stably $\mathbb{Z}$ if $i = 0$ or $n - 1$ and pro-trivial otherwise.* **Note.** *This exercise may be viewed as a continuation of Exercise 3.2.*

5.2. **Siebenmann's thesis.** Theorem 5.3 may be viewed as a classification of those open manifolds that can be compactified to a closed n-ball by addition of an $(n-1)$-sphere boundary. More generally, one may look to characterize open manifolds that can be compactified to a manifold with boundary by addition of a boundary $(n-1)$-manifold. Since the boundary of a manifold P^n always has a *collar neighborhood* $N \approx \partial P^n \times [0,1]$, an open manifold M^n allows such a compactification if and only if it contains a neighborhood of infinity homeomorphic to an *open collar* $Q^{n-1} \times [0,1)$, for some closed $(n-1)$-manifold Q^{n-1}. We refer to open manifolds of this sort as being *collarable*.

The following shows that, to characterize collarable open manifolds, it is not enough to consider the fundamental group at infinity.

Example 20. *Let M^n be the result of a countably infinite collection of copies of $\mathbb{S}^2 \times \mathbb{S}^{n-2}$ connect-summed to $\mathbb{R}^n$ along a sequence of n-balls tending to infinity. See Figure 8. Provided $n \geq 4$, M^n is simply connected at infinity. Moreover, since a compact manifold with boundary has finite homotopy type, and since the addition of a manifold boundary does not affect homotopy type, this M^n admits no such compactification.*

For manifolds that are simply connected at infinity, the necessary additional hypothesis is as simple as one could hope for.

Theorem 5.9 (See [BLL65]). *Let W^n be a 1-ended open n-manifold ($n \geq 6$) that is simply connected at infinity. Then W^n is collarable if and only if $H_* (W; \mathbb{Z})$ is finitely generated.*

For manifolds not simply connected at infinity, the situation is more complicated, but the characterization is still remarkably elegant. It is one of the best-known and most frequently applied theorems in manifold topology.

Theorem 5.10 (Siebenmann's Collaring Theorem). *A 1-ended n-manifold W^n ($n \geq 6$) with compact (possibly empty) boundary is collarable if and only if*

(1) W^n *is inward tame,*
(2) *pro-$\pi_1 (\varepsilon(W^n))$ is stable, and*
(3) $\sigma_\infty (W^n) \in \tilde{K}_0 (\mathbb{Z} [\check{\pi}_1 (\varepsilon (X))])$ *is trivial.*

Remark 8. (a) In the presence of (2), hypotheses (1) and (3) together are equivalent to an assumption that W^n is absolutely inward tame. That allows for a simpler statement of the Collaring Theorem. The power of the stated version is that it allows the finiteness obstruction to be measured on a single (appropriately chosen) neighborhood of infinity. Moreover, in many cases, $\sigma_\infty (W^n)$ is trivial for purely algebraic reasons.

(b) Due to stability, no base ray needs to be mentioned in Condition (2). The use of the Čech fundamental group in Condition (3) is just a convenient way of identifying the single relevant group implied by Condition (2) (see Exercise 4.18).

(c) Since an inward tame manifold with compact boundary is necessarily finite-ended (see Exercise 4.11), it is easy to eliminate the 1-ended hypothesis from the above by requiring that each end satisfy the appropriate conditions.

The proof of Theorem 5.10 is intricate in detail, but simple in concept. We provide a broad outline.

Proof of Siebenmann's Theorem (outline). Begin with a cofinal sequence $\{N_i\}_{i=0}^\infty$ of clean neighborhoods of infinity with $N_{i+1} \subseteq \operatorname{int} N_i$ for all i. After some initial combinatorial group theory, a 2-dimensional disk trading argument allows us to improve the neighborhoods of infinity so that, for each i, N_i and ∂N_i have fundamental groups corresponding to the stable fundamental group $\check{\pi}_1 (\varepsilon(W^n))$. More precisely, each inclusion induces isomorphisms $\pi_1 (\partial N_i) \xrightarrow{\cong} \pi_1 (N_i)$ and $\pi_1 (N_{i+1}) \xrightarrow{\cong} \pi_1 (N_i)$, with each groups being isomorphic to $\check{\pi}_1 (\varepsilon(W^n))$.

Under the assumption that one, and hence all, of the N_i has trivial finiteness obstruction (This utilizes the Sum Theorem for the Wall obstruction, first proved in [Sie65], together with all of the π_1-isomorphisms.), a carefully crafted sequence of modifications to these neighborhoods of infinity—primarily by handle manipulations—results in a further improved sequence of neighborhoods of infinity with the property that $\partial N_i \hookrightarrow N_i$ is a homotopy equivalence for each i. The resulting cobordisms $(A_i, \partial N_i, \partial N_{i+1})$, where $A_i = \overline{N_i - N}_{i+1}$ are then h-cobordisms (See Exercise 5.11).

A clever "infinite swindle" allows one to trivialize the Whitehead torsion of $\partial N_i \hookrightarrow A_i$ in each h-cobordism by inductively borrowing the inverse h-cobordism B_i from

a collar neighborhood of ∂N_{i+1} in A_{i+1} (after which the "new" N_{i+1} is $\overline{N_{i+1} - B_i}$), until the s-cobordism theorem yields $A_i \approx \partial N_i \times [i, i+1]$, for each i. Gluing these products together completes the proof.   □

Exercise 5.11. *Verify the h-cobordism assertion in the above paragraph. In particular, let N_i and N_{i+1} be clean neighborhoods of infinity with int $N_i \supseteq N_{i+1}$ satisfying the properties:* (1) *$\partial N_i \hookrightarrow N_i$ and $\partial N_{i+1} \hookrightarrow N_{i+1}$ are homotopy equivalences and* (2) *$N_{i+1} \hookrightarrow N_i$ induces a π_1-isomorphism. For $A_i = \overline{N_i - N}_{i+1}$, show that both $\partial N_i \hookrightarrow A_i$ and $\partial N_{i+1} \hookrightarrow A_i$ are homotopy equivalences.*

Observe that in the absence of Condition (2), it is still possible to conclude that $(A_i, \partial N_i, \partial N_{i+1})$ is a "1-sided h-cobordism", in particular, $\partial N_i \hookrightarrow A_i$ is a homotopy equivalence.

In the spirit of the result in Exercise 5.4, the following may be obtained as an application of Theorem 5.10.

Theorem 5.12 ([Gui07]). *For an open manifold M^n, the "stabilization" $M^n \times \mathbb{R}$ is collarable if and only if M^n has finite homotopy type. If M^n is finitely dominated, but does not have finite homotopy type, then $M^n \times \mathbb{R}$ satisfies Conditions (1) and (2) of Theorem 5.10, but not Condition (3).*

Sketch of proof. The key step is proving stability of pro-$\pi_1 (M^n \times \mathbb{R}, r)$. We say just enough to convey the main idea—describing a technique that has been useful in several other contexts. Making the argument rigorous is primarily a matter of base points and base rays—a nontrivial issue, but one that we ignore for now.

For simplicity, assume M^n is 1-ended and $N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots$ is a cofinal sequence of clean connected neighborhoods of infinity in M^n. If $R_i = (M^n \times (-\infty, -i] \cup [i, \infty)) \cup (N_i \times \mathbb{R})$, then $\{R_i\}$ forms a cofinal sequence of clean connected neighborhoods of infinity in $M^n \times \mathbb{R}$. If $G = \pi_1 (M^n)$ and $H_i = \mathrm{Im} (\pi_1 (N_i) \to \pi_1 (M^n))$ for each i, then $\pi_1 (R_i) = G *_{H_i} G$ and pro-$\pi_1 (M^n \times \mathbb{R}, r)$ may be represented by

$$G *_{H_0} G \leftarrow G *_{H_1} G \leftarrow G *_{H_2} G \leftarrow \cdots$$

where the bonds are induced by the identities on G factors. Notice that each H_{i+1} injects into H_i. To prove stability, it suffices to show that, eventually, H_{i+1} goes onto H_i. To that end, we argue that every loop in N_i can be homotoped into N_{i+1} by a homotopy whose tracks may go anywhere in M^n.[9] The loops of concern are those lying in $N_i - N_{i+1}$; let α be such a loop, and assume it is an embedded circle.

By the finite domination hypothesis, we may assume the existence of a homotopy K that pulls M^n into $M^n - N_i$. Consider the map $J = K|_{\partial N_i \times [0,1]}$. Adjust J so that it is transverse to the 1-manifold α. Then $J^{-1} (\alpha)$ is a finite collection of circles. With some extra effort we can see that at least one of those circles goes homeomorphically onto α. The strong deformation retraction of $\partial N_i \times [0, 1]$ onto $\partial N_i \times \{0\}$ composed with J pushes α into N_{i+1}.   □

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[9]A complete proof would do this while sliding a basepoint along a base ray r.

Exercise 5.13. *Complete the sketch of the proof of Theorem 5.12 by verifying hypotheses (1) and (3) of Theorem 5.10 under the corresponding assumptions.*

5.3. **Generalizing Siebenmann.** Siebenmann's Collaring Theorem and a "controlled" version of it found in [Qui79] have proven remarkably useful in manifold topology; particularly in obtaining the sorts of structure and embedding theorems that symbolize the tremendous activity in high-dimensional manifold topology in the 1960's and 70's. But the discovery of exotic universal covering spaces, along with a shift in research interests (the Borel and Novikov Conjectures in particular and geometric group theory in general) to topics where an understanding of universal covers is crucial, suggests a need for results applicable to spaces with *non-stable* fundamental group at infinity. As an initial step, one may ask what can be said about open manifolds satisfying some of Siebenmann's conditions—but not π_1-stability. In §4.5 we described a method for constructing locally finite polyhedra satisfying Conditions (1) and (3) of Theorem 5.3, but having almost arbitrary pro-π_1. By the same method, we could build unusual behavior into pro-H_k. So it is a pleasant surprise that, for manifolds with compact boundary, inward tameness by itself, has significant implications.

Theorem 5.14 ([GuTi03, Th.1.2]). *If a manifold with compact (possibly empty) boundary is inward tame, then it has finitely many ends, each of which has semistable fundamental group and stable homology in all dimensions.*

Sketch of proof. Finite-endedness of inward tame manifolds with compact boundary was obtained in Exercise 4.11. The π_1-semistablity of each end is based on the transversality strategy described in Theorem 5.12. Stability of the homology groups is similar, but algebraic tools like duality are also needed. □

Siebenmann's proof of Theorem 5.10 (as outlined above), along with the strategy used by Chapman and Siebenmann in [ChSi76] (to be discussed §8.2) make the following approach all but inevitable. Define a manifold N^n with compact boundary to be a *homotopy collar* if $\partial N^n \hookrightarrow N^n$ is a homotopy equivalence. A homotopy collar is called a *pseudo-collar* if it contains arbitrarily small homotopy collar neighborhoods of infinity. A manifold that contains a pseudo-collar neighborhood of infinity is called *pseudo-collarable.*

Clearly, every collarable manifold is pseudo-collarable, but the Davis manifolds are counterexamples to the converse (see Example 22). Before turning attention to a result like Theorem 5.10, but for pseudo-collarability, we introduce a couple definitions and examine the structure of pseudo-collars.

Recall that a group is *perfect* if its abelianization is the trivial group. A cobordism $(A, \partial_- A, \partial_+ A)$ is called a *one-sided h-cobordism* if $\partial_- A \hookrightarrow A$ is a homotopy equivalence, but not necessarily so for $\partial_+ A \hookrightarrow A$. The key connection between these concepts is provided by:

Lemma 5.15. *Let $(A, \partial_- A, \partial_+ A)$ be a compact one-sided h-cobordism as described above. Then the inclusion $\partial_+ A \hookrightarrow A$ induces $\mathbb{Z}$-homology isomorphisms (in fact,*

$\mathbb{Z}[\pi_1(A)]$-homology) isomorphisms in all dimensions; in addition, $\pi_1(\partial_+ A) \to \pi_1(A)$ is surjective with perfect kernel.

Lemma 5.15 is obtained from various forms of duality. For details, see [GuTi03, Th. 2.5].

Proposition 5.16 (Structure of manifold pseudo-collars). *Let N^n be a pseudo-collar. Then*

(1) *N^n may be expressed as a union $A_0 \cup A_1 \cup A_2 \cup \cdots$ of one-sided h-cobordisms with $\partial_- A_0 = \partial N$ and $\partial_+ A_i = \partial_- A_{i+1} = A_i \cap A_{i+1}$ for all $i \geq 0$,*

(2) *N^n contains arbitrarily small pseudo-collar neighborhoods of infinity,*

(3) *N^n is absolutely inward tame,*

(4) *pro-$H_i(\varepsilon(N^n);\mathbb{Z})$ is stable for all i,*

(5) *pro-$\pi_1(\varepsilon(N^n))$ may be represented by a sequence $G_0 \overset{\mu_1}{\leftarrow} G_1 \overset{\mu_2}{\leftarrow} G_2 \overset{\mu_3}{\leftarrow} \cdots$ of surjections, where each G_i is finitely presentable and each $\ker(\mu_i)$ is perfect, and*

(6) *there exists a proper map $\phi : N^n \to [0,\infty)$ with $\phi^{-1}(0) = \partial N^n$ and $\phi^{-1}(r)$ a closed $(n-1)$-manifold with the same $\mathbb{Z}$-homology as ∂N^n for all r.*

Proof. Observations (1)-(3) are almost immediate, after which (4) and (5) can be obtained by straightforward applications of Lemma 5.15. From there, (6) can be obtained by applying the main result from [DaTi96] to each cobordism $(A_i, \partial_- A_i, \partial_+ A_i)$. $\square$

A couple examples are in order.

Example 21 (The Whitehead manifold is not pseudo-collarable). *First notice that $\mathcal{W}^3$ does contain a homotopy collar neighborhood of infinity. Let D^3 be a tame ball in $\mathcal{W}^3$ and let $N = \mathcal{W}^3 - \mathrm{int}\, D^3$. By excision and the Hurewicz and Whitehead theorems, N is a homotopy collar. (This argument works for all contractible open manifolds.) But since $\mathcal{W}^3$ is neither inward tame nor semistable, Proposition 5.16 assures that $\mathcal{W}^3$ is not pseudo-collarable.*

Example 22 (Davis manifolds are pseudo-collarable). *Non-collarable but pseudo-collarable ends are found in some of our most important examples—the Davis manifolds. It is easy to see that the neighborhood of infinity N_0 shown in Figure 4 is a homotopy collar, as is N_i for each $i > 0$.*

Motivated by Proposition 5.16 and previous definitions, call an inverse sequence of groups *perfectly semistable* if it is pro-isomorphic to an inverse sequence of finitely presentable groups for which the bonding homomorphisms are all surjective with perfect kernels. A complete characterization of pseudocollarable n-manifolds is provided by:

Theorem 5.17 ([GuTi06]). *A 1-ended n-manifold W^n ($n \geq 6$) with compact (possibly empty) boundary is pseudo-collarable if and only if*

(1) *W^n is inward tame,*

(2) *pro-$\pi_1(\varepsilon(W^n))$ is perfectly semistable, and*

(3) $\sigma_\infty (W^n) \in \varprojlim \left\{ \widetilde{K}_0 \mathbb{Z} [\pi_1 (N)] \mid N \ \text{a clean nbd. of infinity} \right\}$ *is trivial.*

Remark 9. In (3), $\sigma_\infty (W_n)$ can be measured as $(\sigma (N_0), \sigma (N_1), \sigma (N_2), \cdots)$, the sequence of Wall finiteness obstructions of an arbitrary nested cofinal sequence of clean neighborhoods of infinity. It is trivial if and only if each N_i has finite homotopy type; so just as in Theorem 5.10, Conditions (1) and (3) together are equivalent to W^n being absolutely inward tame.

By Theorem 5.14, every inward tame open manifold W^n has semistable pro-π_1 and stable pro-H_1. Together those observations guarantee a representation of pro-$\pi_1 (\varepsilon(W^n))$ by an inverse sequence of surjective homomorphisms of finitely presented groups with "nearly perfect" kernels (in a way made precise in [GuTi]). One might hope that Condition (2) of Theorem 5.17 is extraneous, bn example constructed dashes that hope.

Theorem 5.18 ([GuTi03]). *In all dimensions ≥ 6 there exist absolutely inward tame open manifolds that are not pseudo-collarable.*

In light of Theorem 5.17, it is not surprising that Theorem 5.18 uses a significant dose of group theory. In fact, unravelling the group theory at infinity seems to be the key to understanding ends of inward tame manifolds. That topic is the focus of ongoing work [GuTi]. As for our favorite open manifolds, the following is wide-open.

Question 1. *Is the universal cover $\widetilde{M}^n$ of a closed aspherical n-manifold always pseudocollarable? Must it satisfy some of the hypotheses of Theorem 5.17? In particular, is $\widetilde{M}^n$ always inward tame? (If so, an affirmative answer to Conjecture 3 would follow from Theorem 5.14.)*

Again the reader is cautioned that the results in this section rely heavily on manifold-specific tools. For general locally finite complexes, Proposition 4.9 serves as warning. Even so, many ideas and questions discussed in this section have interesting analogs outside manifold topology—in the field of geometric group theory. We take a break from manifold topology to explore that area.

6. End invariants applied to group theory

A standard method for applying topology to group theory is via Eilenberg-MacLane spaces. For a group G, a $K(G, 1)$ *complex* (or *Eilenberg-MacLane complex* for G or a *classifying space* for G) is an aspherical CW complex with fundamental group isomorphic to G. When the language of classifying spaces is used, a $K(G, 1)$ complex is often referred to as an BG complex and its universal cover as an EG complex. Alternatively, an EG complex is a contractible CW complex on which G acts properly and freely.

Exercise 6.1. *Show that a CW complex X is aspherical if and only if $\widetilde{X}$ is contractible.*

It is a standard fact that, for every group G: (a) there exists a $K(G, 1)$ complex, and (b) any two $K(G, 1)$ complexes are homotopy equivalent. Therefore, any homotopy invariant of a $K(G, 1)$ complex is an invariant of G. In that way we define the *(co)homology of G with constant coefficients* in a ring R, denoted $H_*(G; R)$ and $H^*(G; R)$, to be $H_*(K(G, 1); R)$ and $H^*(K(G, 1); R)$, respectively.

At times it is useful to relax the requirement that a BG or an EG be a CW complex. For example, an aspherical manifold or a CAT(0) space with fundamental group G, but with no obvious cell structure might be a used as a BG. Provided the space in question is an ANR, there is no harm in allowing it since all of the key facts from algebraic topology (for example, Exercise 6.1) still apply. Moreover, by Proposition A.2, ANRs are homotopy equivalent to CW complexes, so, if necessary, an appropriate complex can be obtained.

6.1. **Groups of type F.** We say that G *has type F* if $K(G, 1)$ complexes have finite type or, equivalently, there exits a finite $K(G, 1)$ complex or a compact ANR $K(G, 1)$ space. Note that if K is a finite $K(G, 1)$ complex, then $\widetilde{K}$ is locally finite and the G-action is cocompact; we call $\widetilde{K}$ a cocompact EG complex.

Example 23. *All finitely generated free and free abelian groups have type F, as do the fundamental groups of all closed surfaces, except for $\mathbb{R}P^2$. In fact, the fundamental group of every closed aspherical manifold has type F. No group that contains torsion can have type F (see [Geo08, Prop. 7.2.12]), but every torsion-free $CAT(0)$ or δ-hyperbolic group has type F.*

For groups of type F, there is an immediate connection between group theory and topology at the ends of noncompact spaces. If G is nontrivial and K_G is a finite $K(G, 1)$ complex, $\widetilde{K}_G$ is contractible, locally finite, and noncompact, and by Corollary 3.10, all other finite $K(G, 1)$ complexes (or compact ANR classifying spaces) have universal covers proper homotopy equivalent to $\widetilde{K}_G$. So the end invariants of $\widetilde{K}_G$, which are well-defined up to proper homotopy equivalence, may be attributed directly to G. For example, one may discuss: the *number of ends* of G; the homology at infinity of G (pro-$H_*\varepsilon(G); R)$ or $\check{H}_*(\varepsilon(G); R)$); and the homotopy behavior of the end(s) of G—properties such as simple connectedness, stability, semistability, or pro-monomorpic at infinity. In cases where $\widetilde{K}_G$ is 1-ended and semistable, pro-$\pi_*(\varepsilon(G))$ and $\check{\pi}_*(\varepsilon(G))$ are also group invariants. The need for semistability is, of course, due to base ray issues. Although $\widetilde{K}_G$ is well-defined up to proper homotopy type, there is no canonical choice base ray; semistability makes that issue go away. More on that topic shortly.

6.2. **Groups of type F_k.** In fact, the existence of a finite $K(G, 1)$ is excessive for defining end invariants like pro-$H_*(\varepsilon(G); R))$ and $\check{H}_*(\varepsilon(G); R)$. If G admits a $K(G, 1)$ complex K with a finite k-skeleton (in which case we say G *has type F_k*), then all j-dimensional homology and homotopy end properties of the (locally finite) k-skeleton $\widetilde{K}_G^{(k)}$ of $\widetilde{K}_G$ can be directly attributed to G, provided $j < k$. The proof of invariance is rather intuitive. If L is any other $K(G, 1)$ with finite k-skeleton, choose

a cellular homotopy equivalence $f : K \to L$ and a homotopy inverse $g : L \to K$. These lift to homotopy equivalences $\tilde{f} : \widetilde{K} \to \widetilde{L}$ and $\tilde{g} : \widetilde{L} \to \widetilde{K}$, which cannot be expected to be proper. Nevertheless, the restrictions of $\tilde{g} \circ \tilde{f}$ and $\tilde{f} \circ \tilde{g}$ to the $(k-1)$-skeletons of $\widetilde{K}$ and $\widetilde{L}$ can be proven properly homotopic to inclusions $\widetilde{K}^{(k-1)} \hookrightarrow \widetilde{K}^{(k)}$ and $\widetilde{L}^{(k-1)} \hookrightarrow \widetilde{L}^{(k)}$. This is enough for the desired result.

As an example of the above, the number of ends, viewed as $\left| \check{\pi}_0 \left(\widetilde{K}_G^{(1)} \right) \right|$ is a well-defined invariant of a finitely generated groups, ie, groups of type F_1.

Exercise 6.2. *Equivalently, one may define the number of ends of a finitely generated G to be the number of ends of a corresponding Cayley graph. Explain why this definition is equivalent to the above.*

There are important connections between pro-$H_* \left(\varepsilon \left(G \right); R \right);)$ and $\check{H}_* \left(\varepsilon \left(G \right); R \right)$ and the group homology and cohomology of G, as presented in [Bro94]. We have chosen not to delve into that topic. The interested reader is encouraged to read Chapters 8 and 13 of [Geo08].

6.3. **Ends of Groups.** In view of the above comments, the following iconic result may be viewed as an application of $\check{\pi}_0 \left(\varepsilon \left(G \right) \right)$.

Theorem 6.3 (Freudenthal-Hopf-Stallings)**.** *Every finitely generated group G has $0,1,2$, or infinitely many ends. Moreover*

    (1) *G is 0-ended if and only if it is finite,*
    (2) *G is 2-ended if and only if it contains an infinite cyclic group of finite index, and*
    (3) *G is infinite-ended if and only if*
        (a) *$G = A *_C B$, where C is finite and has index ≥ 2 in both A and B with at least one index being ≥ 3, or*
        (b) *$G = A*_\phi$ where ϕ is an isomorphism between finite subgroups of G each having index ≥ 2 in G.*

Proof (small portions). The opening line of Theorem 6.3 is essentially Exercise 3.1; item (1) is trivial and item (2) is a challenging exercise. Item (3) is substantial [Sta68], but pleasantly topological. Complete treatments can be found in [ScWa79] or [Geo08]. □

6.4. **The Semistability Conjectures.** If G is finitely presentable, ie, G has type F_2, and K is a corresponding presentation 2-complex (or any finite complex with fundamental group G), then K may be realized as the 2-skeleton of a $K(G, 1)$. Hence, pro-$H_1 \left(\varepsilon \left(\widetilde{K} \right); R \right)$ and $\check{H}_1 \left(\varepsilon \left(\widetilde{K} \right); R \right)$ are well-defined invariants of G. Similarly, when G is 1-ended, properties such as simple connectivity, stability, semistability and pro-monomorphic at infinity can be attributed to G, itself. In an effort to go further with homotopy properties of the end of G, we are inexorably led back to the open problem:

Conjecture 4 (Semistability Conjecture–with explanation)**.** *Every 1-ended finitely presented group G is semistable. In other words, the universal cover $\widetilde{K}$ of every finite complex with fundamental group G is strongly connected at infinity; equivalently, pro-$\pi_1\left(\widetilde{K}, r\right)$ semistable for some (hence all) proper rays r.*

The fundamental nature of the Semistability Conjecture should now be clear. One would like to view pro-$\pi_1\left(\varepsilon\left(\widetilde{K}\right); r\right)$ and $\check{\pi}_1\left(\varepsilon\left(\widetilde{K}\right); r\right)$ as invariants of G. Unfortunately, there is a the potential for these invariants to depend upon base rays. A positive resolution of the Semistability Conjecture would eliminate that complication. The same applies to pro-π_j and $\check{\pi}_j$ when G is of type F_k and $j < k$.

The generalization of Conjecture 4 to groups with arbitrary numbers of ends is also valid, but simpler than one might expect. For 0-ended groups there is nothing to discuss, all 2-ended groups are known to be simply connected at each end. (See Exercise 6.4 below.), and Mihalik [Mih87] has shown that an affirmative answer for 1-ended groups would imply an affirmative answer for all infinite-ended groups.

Exercise 6.4. *Let G be a group of type F_k. Show that every finite index subgroup H is of type F_k and the two groups share the same end invariants through dimension $k-1$. Conclude that every 2-ended group is simply connected at each end.*

Evidence for the Semistability Conjecture is provided by a wide variety of special cases; here is a sampling.

Theorem 6.5. *A finitely presented group satisfying any one of the following is semistable.*

(1) *G is the extension of an infinite group by an infinite group,*
(2) *G is a one-relator group,*
(3) *$G = A *_C B$ where A and B are finitely presented and semistable and C is infinite,*
(4) *$G = A*_C$ where A is finitely generated and semistable and C is infinite,*
(5) *G is δ-hyperbolic,*
(6) *G is a Coxeter group,*
(7) *G is an Artin group.*

References include: [Mih83], [MiT92a], [MiT92b], [Swa96], and [Mih96].

There is a variation on the Semistability Conjecture that is also open.

Conjecture 5 (H_1-semistability Conjecture)**.** *For every 1-ended finitely presented group G, pro-$H_1(G; \mathbb{Z})$ is semistable.*

Since pro-$H_1(\varepsilon(G); \mathbb{Z})$ can be obtained by abelianization of pro-$\pi_1(\varepsilon(G))$, it is clear that the H_1-semistability Conjecture is weaker than the Semistability Conjecture. Moreover, the H_1-version of one of our favorite special cases of the Semistability Conjecture—the case where G is the fundamental group of an aspherical manifold—is easily solved in the affirmative, by an application of Exercise 5.8. This provides a ray of hope that the Manifold Semistability Conjecture is more accessible that the general case.
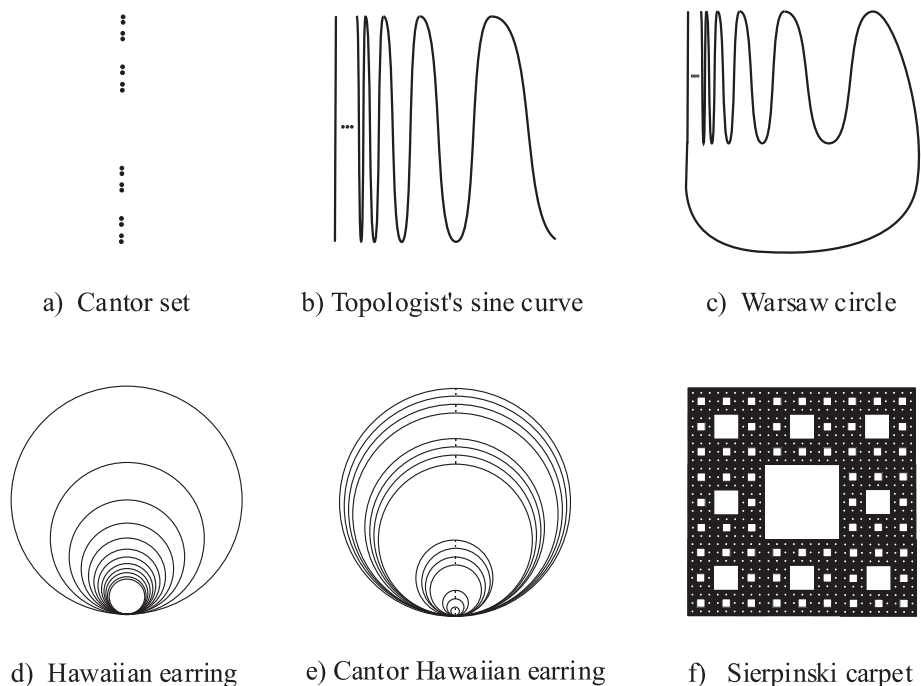
a) Cantor set b) Topologist's sine curve c) Warsaw circle

d) Hawaiian earring e) Cantor Hawaiian earring f) Sierpinski carpet

FIGURE 9.

7. SHAPE THEORY

Shape theory may be viewed as a method for studying bad spaces using tools created for the study of good spaces. Although more general approaches exist, we follow the classical (and most intuitive) route by developing shape theory only for compacta. But now we are interested in arbitrary compacta—not just ANRs. A few examples to be considered are shown in Figure 9.

The abrupt shift from noncompact spaces with nice local properties to compacta with bad local properties may seem odd, but there are good reasons for this (temporary) shift in focus. First, the tools we have already developed for analyzing the ends of manifolds and complexes are nearly identical to those used in shape theory; understanding and appreciating the basics of shape theory will now be quite easy. More importantly, certain aspects of the study of ends are nearly impossible without shapes—if the theory did not already exist, we would be forced to invent it.

For more comprehensive treatments of shape theory, the reader can consult [Bor75] or [DySe78].

7.1. **Associated sequences, basic definitions, and examples.** In shape theory, the first step in studying a compactum A is to choose an *associated inverse sequence* $K_0 \xleftarrow{f_1} K_1 \xleftarrow{f_2} K_2 \xleftarrow{f_3} \cdots$ of finite polyhedra and simplicial maps. There are several ways this can be done. We describe a few of them.

Method 1: If A is finite-dimensional, choose an embedding $A \hookrightarrow \mathbb{R}^n$, and let $K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots$ be a sequence of compact polyhedral neighborhoods intersecting in

A. Since it will be impossible to choose triangulations under which all of the inclusion maps are be simplicial, just choose progressively finer triangulations for the K_i and let the f_i be simplicial approximations to the inclusion maps.

Method 2: Choose a sequence $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \cdots$ of finite covers of A by ε_i-balls, where $\varepsilon_i \to 0$ and each $\mathcal{U}_{i+1}$ refines $\mathcal{U}_i$. Let the K_i be nerves of the $\mathcal{U}_i$ and choose the f_i to be simplicial maps that take each vertex $U \in \mathcal{U}_i$ to a vertex $V \in \mathcal{U}_{i-1}$ with $U \subseteq V$.

Method 3: If A can be expressed as the inverse limit of an inverse sequence $K_0 \xleftarrow{g_1} K_1 \xleftarrow{g_2} K_2 \xleftarrow{g_3} \cdots$ of finite polyhedra, then that sequence itself may be associated to A, after each map is approximated by one that is simplicial.

Remark 10. (a) At times, it will be convenient if each K_i in an associated inverse sequence has a preferred vertex p_i with each f_{i+1} taking p_{i+1} to p_i. That can easily be arranged; we refer to the result as a *pointed inverse sequence.*

(b) Our requirement that the bonding maps in associated inverse sequences be simplicial, will soon be seen as unnecessary. But, for now, there is no harm in including that additional niceness condition.

(c) When A is infinite-dimensional, a variation on Method 1 is available. In that case, A is embedded in the Hilbert cube and a sequence $\{N_i\}$ of compact Hilbert cube neighborhoods of A is chosen. Since each N_i has the homotopy type of a finite polyhedron K_i, an associated inverse sequence for A is easily obtained.

The choice of an associated inverse sequence for a compactum A should be compared to the process of choosing a cofinal sequence of neighborhoods of infinity for a noncompact space X. In both situations, the terms in the sequences can be viewed as progressively better approximations to the object of interest, and in both situations, there is tremendous leeway in assembling those approximating sequences. In both contexts, that leeway raises well-definedness issues. In the study of ends, we used an equivalence relation based on ladder diagrams to obtain the appropriate level of well-definedness. The same is true in shape theory.

Proposition 7.1. *For a fixed compactum A, let $\{K_i, f_i\}$ and $\{L_i, g_i\}$ be a pair of associated inverse sequences of finite polyhedra. Then there exist subsequences, simplicial maps, and a corresponding ladder diagram*

$$K_{i_0} \xleftarrow{f_{i_0+1,i_1}} K_{i_1} \xleftarrow{f_{i_1+1,i_2}} K_{i_2} \xleftarrow{f_{i_2+1,i_3}} K_{i_3} \quad \cdots$$

$$L_{j_0} \xleftarrow{g_{j_0+1,j_1}} L_{j_1} \xleftarrow{g_{j_1+1,j_2}} L_{j_2} \xleftarrow{g_{j_2+1,j_3}} \quad \cdots$$

in which each triangle of maps homotopy commutes. If desired, we may require that those homotopies preserve base points.

Exercise 7.2. *Prove some or all of Proposition 7.1. Start by comparing any pair sequences obtained using the same method. then note that Method 1 is a special case of Method 3.*

Define a pair of inverse sequences of finite polyhedra and simplicial maps to be pro-*homotopy equivalent* if they contain subsequences that fit into a homotopy commuting ladder diagram, as described in Proposition 7.1. Compacta A and A' are *shape equivalent* if some (and thus every) pair of associated inverse sequences of finite polyhedra are pro-homotopy equivalent. In that case we write $\mathcal{S}h(A) = \mathcal{S}h(A')$ or sometimes $A' \in \mathcal{S}h(A)$.

Remark 11. If $\{K_i, f_i\}$ is an associated inverse sequence for a compactum A, it is not necessarily the case that $\varprojlim \{K_i, f_i\} \approx A$. But it is immediate from the definitions that the two spaces have the same shape.

Exercise 7.3. *Show that the Topologist's Sine Curve has the shape of a point and the Warsaw Circle has the shape of a circle. Note that neither space is homotopy equivalent to its nicer shape version.*

Exercise 7.4. *Show that the Whitehead Continuum (see Example 1) has the shape of a point. Spaces with the shape of a point are often called* cell-like.

Exercise 7.5. *Show that the Sierpinski Carpet is shape equivalent to a Hawaiian Earring.*

Exercise 7.6. *Show that the Cantor Hawaiian Earring is shape equivalent to a standard Hawaiian Earring. (An observation that once prompted the reaction: "I demand a recount!")*

When considering the shape of a compactum A, the space A itself becomes largely irrelevant after an associated inverse sequence has been chosen. In a sense, shape theory is just the study of pro-homotopy classes of inverse sequences of finite polyhedra. Nevertheless, there is a strong correspondence between inverse sequences of finite polyhedra and compact metric spaces themselves. If A is the inverse limit of an inverse sequence $\{K_i, f_i\}$ of finite polyhedra, then applying any of the three methods mentioned earlier to the space A yields an inverse sequence of finite polyhedra pro-homotopy equivalent to the original $\{K_i, f_i\}$. In other words, passage to an inverse limit preserves all relevant information. As we saw in Exercise 4.3, that is not the case with inverse sequences of groups. This phenomenon is even more striking when studying ends of spaces. If $N_0 \hookleftarrow N_1 \hookleftarrow N_2 \hookleftarrow \cdots$ is a cofinal sequence of neighborhoods of infinity of a space X, the inverse limit of that sequence is clearly the empty set. In some sense, the study of ends is a study of an imaginary "space at infinity". By using shape theory, we can sometimes make that space a reality.

Exercise 7.7. *Prove that an inverse sequence of nonempty finite polyhedra (more generally, an inverse sequence of nonempty compacta) is never the empty set.*

Exercise 7.8. *So far, our discussion of shape has focused on exotic compacta; but nice spaces, such as finite polyhedra, are also part of the theory. Show that finite polyhedra K and L are shape equivalent if and only if they are homotopy equivalent. Hint. Choosing trivial associated inverse sequences $K \xleftarrow{\mathrm{id}} K \xleftarrow{\mathrm{id}} \cdots$ and $L \xleftarrow{\mathrm{id}} L \xleftarrow{\mathrm{id}} \cdots$ makes the task easier. A more general observation of this sort will stated shortly.*

7.2. **The algebraic shape invariants.** In the spirit of the work already done on ends of spaces, we define a variety of algebraic invariants for compacta. Given a compactum A and any associated inverse sequence $\{K_i, f_i\}$, define pro-$H_*(A; R)$ to be the pro-isomorphism class of the inverse sequence

$$H_*(K_0; R) \xleftarrow{\lambda_1} H_*(K_1; R) \xleftarrow{\lambda_2} H_*(K_2; R) \xleftarrow{\lambda_3} \cdots$$

and $\check{H}_*(A; R)$ to be its inverse limit. By reversing arrows and taking a direct limit, we also define ind-$H^*(A; R)$ and $\check{H}^*(A; R)$. The groups $\check{H}_*(A; R)$ and $\check{H}^*(A; R)$ are know as the *Čech homology and cohomology groups* of A, respectively. If we begin with a pointed inverse sequence $\{(K_i, p_i), f_i\}$ we obtain pro-$\pi_*(A, p)$ and $\check{\pi}_*(A, p)$, where p corresponds to $(p_0, p_1, p_2, \cdots)$. Call $\check{\pi}_*(A, p)$ the *Čech homotopy groups* of A, or sometimes, the *shape groups* of A.

Čech cohomology is known to be better-behaved than Čech homology, in that there is a full-blown Čech cohomology theory satisfying the Eilenberg-Steenrod axioms. Although the Čech homology groups of A do not fit into such a theory, they are are still perfectly good algebraic invariants of of A. As we have seen in the past, pro-$H_*(A; R)$ and pro-$\pi_*(A, p)$ tend to carry more information than the corresponding inverse limits.

Exercise 7.9. *Observe that, for the Warsaw circle W, the first Čech homology and the first Čech homotopy group are not the same as the first singular homology and traditional fundamental group of W.*

7.3. **Relaxing the definitions.** Now that the framework for shape theory is in place, we make a few adjustments to the definitions. These changes will not nullify anything done so far, but at times they will make the application of shape theory significantly easier.

Previously we required bonding maps in associated inverse sequences to be simplicial. That has some advantages; for example, one can define the pro- and Čech homology of A using only simplicial homology. But in light of the definition of pro-homotopy equivalence, it is clear that only the homotopy classes of the bonding maps matters. So, adjusting a naturally occurring bonding map to make it simplicial is unnecessary. From now on, we only require bonding maps to be continuous. In a similar vein, a finite polyhedron K_i in an inverse sequence corresponding to A might be replaced by another (nice) space with the same homotopy type as K_i. For example, we will allow arbitrary finite CW complexes. More generally, any compact ANR is acceptable since it is homotopy equivalent to a finite complex (Proposition A.2).

With the above relaxation of definitions in place, the following fundamental facts becomes elementary.

Proposition 7.10. *Let A and B be compact ANRs. Then $\mathcal{S}\mathrm{h}(A) = \mathcal{S}\mathrm{h}(B)$ if and only if $A \simeq B$.*

Proof. The argument used in Exercise 7.8 can be applied here.   □

Proposition 7.11. *If A is a compactum that is shape equivalent to a compact ANR B, then* pro-$H_*(A; R)$ *and* pro-$\pi_*(A, p)$ *are stable for all $*$ with $\check{H}_*(A; R) = H_*(B; R)$*

and $\check{\pi}_* (A, p) = \pi_* (B, p)$. *In particular,* $\check{H}_* (A; R)$ *is finitely generated, for all* $*$ *and* $\check{\pi}_1 (A, p)$ *is finitely presentable.*

Proof. Choose the trivial associated inverse sequence $B \xleftarrow{\mathrm{id}} B \xleftarrow{\mathrm{id}} \cdots$. $\qquad\square$

Example 24. *Compacta d), e), and f) from Figure 9 do not have the shapes of ANRs.*

Let *pro-$\mathcal{H}$omotopy* denote the set of all pro-homotopy classes of inverse sequences of compact ANRs and continuous maps, where the equivalence relation is determined as in Proposition 7.1. If $\mathcal{S}hapes$ denotes the set of all shape classes of compact metric spaces, then there is a natural bijection $\Theta : pro\text{-}\mathcal{H}omotopy \to \mathcal{S}hapes$ defined by taking inverse limits; Methods 1-3 above determine Θ^{-1}. With some additional work we could define morphisms in *pro-$\mathcal{H}$omotopy* as certain equivalence classes of sequences of maps, thereby promoting *pro-$\mathcal{H}$omotopy* to a full-fledged category. We could then use Θ to indirectly define morphisms in $\mathcal{S}hapes$ to obtain the shape category. In that case, each continuous function $f : A \to B$ between compacta determines a unique shape morphism; but unfortunately, not every shape morphism from A to B can be realized by a continuous map. This is not as surprising as it first appears—as an example, the reader can attempt to construct a map from $\mathbb{S}^1$ to the Warsaw circle that deserves to be called a shape equivalence.

Exercise 7.12. *Show that homotopy equivalent compacta are shape equivalent.* Warning. *This exercise requires some proficiency with ANR theory. Non-experts may wish to consult Appendix A.*

Remark 12. In order to present a thorough development of the *pro-Homotopy* and $\mathcal{S}hapes$ categories, more care would be required in dealing with base points. In fact, we would end up building slightly different categories in each case—one incorporating base points and the other without base points. When that is done, the equivalence relations on the objects do not change (for example, compacta are shape equivalent if and only if they are "pointed shape equivalent"), but the categories differ in their morphisms. In the context of these notes, we need not be concerned with that distinction.

7.4. **The shape of the end of an inward tame space.** The relationship between shape theory and the topology of the ends of noncompact spaces goes beyond a similarity between the tools used in their studies. In this section we develop a precise relationship between shapes of compacta and ends of inward tame ANRs.

Let Y be a inward tame ANR. By repeated application of the definition of inward tameness, there exist sequences of neighborhoods of infinity $\{N_i\}_{i=0}^{\infty}$, finite complexes $\{K_i\}_{i=1}^{\infty}$, and maps $f_i : N_i \to K_i$ and $g_i : K_i \to N_{i-1}$ with $g_i f_i \simeq \mathrm{incl}\,(N_i \hookrightarrow N_{i-1})$ for all i. By letting $h_i = f_{i-1} g_i$, these can be assembled into a homotopy commuting ladder diagram

The pro-homotopy equivalence class of $K_1 \xleftarrow{h_2} K_2 \xleftarrow{h_3} K_3 \xleftarrow{h_4} \cdots$ is fully determined by Y. That is easily verified by a diagram of the form (4.2) along with transitivity. Define the *shape of the end of* Y, denoted $\mathcal{S}h(\varepsilon(Y))$, to be the shape class of $\varprojlim \{K_i, h_i\}$. A compactum $A \in \mathcal{S}h(\varepsilon(Y))$ can be viewed as a physical representative of the illusive "end of Y".

The following is immediate.

Theorem 7.13. *Let Y be an inward tame ANR and $A \in \mathcal{S}h(\varepsilon(Y))$. Then*

(1) pro-$H_i(\varepsilon(Y); R) = $ pro-$H_i(A; R)$ *and* $\check{H}_i(\varepsilon(Y); R) \cong \check{H}_i(A; R)$ *for all i and any coefficient ring R, and*

(2) *if Y is 1-ended and semistable then* pro-$\pi_i(\varepsilon(Y)) = $ pro-$\pi_i(A)$ *and* $\check{\pi}_i(\varepsilon(Y)) \cong \check{\pi}_i(A)$ *for all i.*

The existence of diagrams like (4.4) shows that $\mathcal{S}h(\varepsilon(Y))$ is also an invariant of the proper homotopy class of Y. There is also a partial converse to that statement—an assertion about the proper homotopy type of Y based only on the shape of its end. Since the topology at the end of a space does not determine the global homotopy type of that space, a new definition is required.

Spaces X and Y are *homeomorphic at infinity* if there exists a homeomorphism $h : N \to M$, where $N \subseteq X$ and $M \subseteq Y$ are neighborhoods of infinity. They are *proper homotopy equivalent at infinity* if there exist neighborhoods of infinity $N' \subseteq N$ in X and $M' \subseteq M$ in Y and proper maps $f : N \to Y$ and $g : M \to X$, with $g \circ f|_{N'} \stackrel{p}{\simeq} \mathrm{incl}(N' \hookrightarrow X)$ and $f \circ g|_{M'} \stackrel{p}{\simeq} \mathrm{incl}(M' \hookrightarrow Y)$.

Theorem 7.14. *Let X and Y be inward tame ANRs. Then $\mathcal{S}h(\varepsilon(X)) = \mathcal{S}h(\varepsilon(Y))$ if and only if X and Y are proper homotopy equivalent at infinity.*

Proof. The reverse implication follows from the previous paragraphs, while the forward direction is nontrivial. A proof can be can be obtained by combining results from [ChSi76] and [EdHa76]. $\square$

In certain circumstances, the "at infinity" phrase can be removed from the above. For example, we have.

Corollary 7.15. *Let X and Y be contractible inward tame ANRs. Then $\mathcal{S}h(\varepsilon(X)) = \mathcal{S}h(\varepsilon(Y))$ if and only if X and Y are proper homotopy equivalent.*

Exercise 7.16. *Use the Homotopy Extension Property to obtain Corollary 7.15 from Theorem 7.14.*

Example 25. *If $K_0 \xleftarrow{f_1} K_1 \xleftarrow{f_2} K_2 \xleftarrow{f_3} \cdots$ is a sequence of finite complexes and $A = \varprojlim \{K_i, f_i\}$, it is easy to see that A represents $\mathcal{S}h(\varepsilon(\mathrm{Tel}(\{K_i, f_i\})))$. By Theorem 7.14, any inward tame space X with $\mathcal{S}h(\varepsilon(X)) = A$ is proper homotopy equivalent at infinity to X; and if X is contractible, $X \stackrel{p}{\simeq} \mathrm{CTel}(\{K_i, f_i\})$. In some sense, the inverse mapping telescope provides an uncomplicated model of the end behavior of any inward tame space.*

Remark 13. In this section, we have intentionally not required spaces to be 1-ended. So, for example, $\mathcal{S}\mathrm{h}(\varepsilon(\mathbb{R}))$ is representable by a 2-point space and the shape of the end of a ternary tree is representable by a Cantor set. For more complex multi-ended X, individual components of $A \in \mathcal{S}\mathrm{h}(\varepsilon(X))$ may have nontrivial shapes, and to each end of X there will be a component of A whose shape reflects properties of that end.

8. $\mathcal{Z}$-SETS AND $\mathcal{Z}$-COMPACTIFICATIONS

While reading §7.4, the following thought may have occurred to the reader: *For inward tame X with $\mathcal{S}\mathrm{h}(\varepsilon(X)) = A$, is there a way to glue A to the end of X to obtain a nice compactification?* As stated, that question is a bit too simple, but it provides reasonable motivation for the coming sections.

8.1. **Definitions and examples.** A closed subset A of an ANR X is a $\mathcal{Z}$-*set* if either of the following equivalent conditions is satisfied:

- There exists a homotopy $H : X \times [0,1] \to X$ such that $H_0 = \mathrm{id}_X$ and $H_t(X) \subseteq X - A$ for all $t > 0$. (We say that H *instantly homotopes X off from A*.)
- For every open set U in X, $U - A \hookrightarrow U$ is a homotopy equivalence.

The second condition explains some alternative terminology: $\mathcal{Z}$-sets are sometimes called *homotopy negligible* sets.

Example 26. *The $\mathcal{Z}$-sets in a manifold M^n are precisely the closed subsets of ∂M^n. In particular, ∂M^n is a $\mathcal{Z}$-set in M^n.*

Example 27. *It is a standard fact that every compactum A can be embedded in the Hilbert cube $\mathcal{Q}$. It may be embedded as a $\mathcal{Z}$-set as follows: embed A in the "face" $\{1\} \times \prod_{i=2}^{\infty} [-1,1] \subseteq \prod_{i=1}^{\infty} [-1,1] = \mathcal{Q}$.*

Example 27 is the starting point for a remarkable characterization of shape, sometimes used as an alternative definition. We will not attempt to describe a proof.

Theorem 8.1 (Chapman's Complement Theorem)**.** [Cha76] *Let A and B be compacta embedded as $\mathcal{Z}$-sets in $\mathcal{Q}$. Then $Sh(A) = Sh(B)$ if and only if $\mathcal{Q} - A \approx \mathcal{Q} - B$*

A $\mathcal{Z}$-*compactification* of a space Y is a compactification $\overline{Y} = Y \sqcup Z$ with the property that Z is a $\mathcal{Z}$-set in $\overline{Y}$. In this case, Z is called a $\mathcal{Z}$-*boundary* for Y. Implicit in this definition is the requirement that $\overline{Y}$ be an ANR; moreover, since an open subset of an ANR is an ANR, Y itself must be an ANR to be a candidate for $\mathcal{Z}$-compactification. A result from the ANR theory ensures that every compactification $\overline{Y}$ of an ANR Y, for which $\overline{Y} - Y$ satisfies either of the negligibility conditions in the definition of $\mathcal{Z}$-set, is necessarily an ANR; hence, it is a $\mathcal{Z}$-compactification.

A nice property of a $\mathcal{Z}$-compactification is that homotopy type is unchanged; for example, a $\mathcal{Z}$-compactification of a contractible space is contractible. The prototypical example is the compactification of $\mathbb{R}^n$ to an n-ball by addition of the $(n-1)$-sphere at infinity; the prototypical non-example is the 1-point compactification of $\mathbb{R}^n$. Finer relationships between Y, $\overline{Y}$, and Z can be understood via shape theory and the study of ends. Before moving in that direction, we add to our collection of examples.

Example 28. *In manifold topology, the most fundamental $\mathcal{Z}$-compactification is the addition of a manifold boundary to an open manifold, as discussed in §5.2.*

Not all $\mathcal{Z}$-compactifications of open manifolds are as simple as the above.

Example 29. *Let C^n be a Newman contractible n-manifold embedded in $\mathbb{S}^n$ (as it is by construction). A non-standard $\mathcal{Z}$-compactification of $\operatorname{int}\mathbb{B}^{n+1}$ can be obtained by crushing C^n to a point. In this case, the quotient $\mathbb{S}^n/C^n$ is a $\mathcal{Z}$-set in $\mathbb{B}^{n+1}/C^n$. Note that $\mathbb{S}^n/C^n$ is not a manifold!*

For those who prefer lower-dimensional examples, a similar $\mathcal{Z}$-compactification of $\operatorname{int}\mathbb{B}^4$ can be obtained by crushing out a wild arc or a Whitehead continuum in $\mathbb{S}^3$. In terms of dimension, that is as low as it gets. As a result of Corollary 10.4 (still to come) , for $n \leq 2$, a $\mathcal{Z}$-boundary of $\mathbb{B}^{n+1}$ is necessarily homeomorphic to $\mathbb{S}^n$.

Example 30. *Let ΣC^n be the suspension of a Newman compact contractible n-manifold. The suspension of ∂C^n is a $\mathcal{Z}$-set in ΣC^n, and its complement, $\operatorname{int} C^n \times (-1,1)$, is homeomorphic to $\mathbb{R}^{n+1}$ by Exercise 5.4. So this is another nonstandard $\mathcal{Z}$-compactification of $\mathbb{R}^{n+1}$.*

Exercise 8.2. *Verify the assertions made in Examples 29 and 30.*

Often a manifold that cannot be compactified by addition of a manifold boundary is, nevertheless, $\mathcal{Z}$-compactifiable—a fact that is key to the usefulness of $\mathcal{Z}$-compactifications. Davis manifolds are the ideal examples.

Example 31. *The 1-point compactification of the infinite boundary connected sum $C_0^n \overset{\partial}{\#} (-C_1^n) \overset{\partial}{\#} (C_2^n) \overset{\partial}{\#} (-C_3^n) \overset{\partial}{\#} \cdots$ shown at the top of Figure 3 is a $\mathcal{Z}$-compactification. More significantly the point at infinity together with the original manifold boundary form a $\mathcal{Z}$-boundary for the corresponding Davis manifold $\mathcal{D}^n$. It is interesting to note that $\mathcal{D}^n$ cannot admit a $\mathcal{Z}$-compactification with boundary a manifold (or even an ANR) since pro-π_1 is not stable. This will be explained soon.*

Example 32. *In geometric group theory, the prototypical $\mathcal{Z}$-compactification is the addition of the visual boundary $\partial_\infty X$ to a proper CAT(0) space X. Indeed, if $\partial_\infty X$ is viewed as the set of end points of all infinite geodesic rays emanating from a fixed $p_0 \in X$, a homotopy pushing inward along those rays verifies the $\mathcal{Z}$-set property.*

Example 33. *In [ADG97], an equivariant CAT(0) metric is placed on many of the original Davis manifolds. In [DaJa91] an entirely different construction produces locally CAT(0) closed aspherical manifolds, whose CAT(0) universal covers are similar to Davis' earlier examples. These objects with their $\mathcal{Z}$-compactifications and $\mathcal{Z}$-boundaries provide interesting common ground for manifold topology and geometric group theory.*

At the expense of losing the isometric group action, Fischer and Guilbault [FiGu05] have placed CAT(−1) metrics on the Davis manifolds in such a way that the visual sphere at infinity is a homogeneous and nowhere locally contractible. Their method can also be used to place CAT(−1) metrics on the asymmetric Davis manifolds from Example 4.

Example 34. *If* $K_0 \xleftarrow{f_1} K_1 \xleftarrow{f_2} K_2 \xleftarrow{f_3} \cdots$ *is an inverse sequence of finite polyhe-dra (or finite CW complexes or compact ANRs), then the inverse mapping telescope* $\mathrm{Tel}\left(\{K_i, f_i\}\right)$ *can be* $\mathcal{Z}$*-compactified by adding a copy of* $\varprojlim \{K_i, f_i\}$, *a space that contains one point for each of the infinite telescope rays described in §4.5. Note the similarity of this to Example 32.*

In §9 and 10, we will look at applications of $\mathcal{Z}$-compactification to geometric group theory and manifold topology. Before that, we address a pair of purely topological questions:

- *When is a space* $\mathcal{Z}$*-compactifiable?*
- *To what extent are* $\mathcal{Z}$*-boundaries unique?* (Examples 29 and 30 show that a space can admit nonhomeomorphic $\mathcal{Z}$-boundaries.)

8.2. **Existence and uniqueness for** $\mathcal{Z}$**-compactifications and** $\mathcal{Z}$**-boundaries.**
If Y admits a $\mathcal{Z}$-compactification $\overline{Y} = Y \sqcup Z$, then as noted above, Y must be an ANR; and since $Y \hookrightarrow \overline{Y}$ is a homotopy equivalence and $\overline{Y}$ is a compact ANR, Proposition A.2 implies that Y has finite type. Prying a bit deeper, a homotopy $H : \overline{Y} \times [0, 1] \to \overline{Y}$ that instantly homotopes $\overline{Y}$ off Z can be "truncated" (with the help of the Homotopy Extension Property) to homotope arbitrarily small closed neighborhoods of infinity (in Y) into compact subsets. Hence, Y is necessarily inward tame.

By combining the results noted in Examples 25 and 34, every inward tame ANR is proper homotopy equivalent to one that is $\mathcal{Z}$-compactifiable. Unfortunately, $\mathcal{Z}$-compactifiability is not an invariant of proper homotopy type. The following result begins to make that clear.

Proposition 8.3. *Every* $\mathcal{Z}$*-compactifiable space that is sharp at infinity is absolutely inward tame.*

Proof. If $\overline{Y} = Y \sqcup Z$ is a $\mathcal{Z}$-compactification and N is a closed ANR neighborhood of infinity in Y. Then $\overline{N} \doteq N \sqcup A$ is $\mathcal{Z}$-compactification of N, hence a compact ANR, and therefore homotopy equivalent to a finite complex K. Since $N \hookrightarrow \overline{N}$ is a homotopy equivalence, $N \simeq K$. $\square$

Remark 14. By employing the standard trick of considering $X \times \mathcal{Q}$ (to ensure sharpness at infinity), Proposition 8.3 provides an alternative proof that a $\mathcal{Z}$-compactifiable ANR must be inward tame.

Theorem 8.4. *Suppose Y admits a* $\mathcal{Z}$*-compactification $\overline{Y} = Y \sqcup Z$. Then $Z \in \mathcal{S}\mathrm{h}(\varepsilon(Y))$.*

Proof. Arguing as in Remark 14, we may assume without loss of generality that Y is sharp at infinity. Choose a cofinal sequence $\{N_i\}$ of closed ANR neighborhoods of infinity in Y, and for each i let $\overline{N}_i$ be the compact ANR $N_i \sqcup Z$. The homotopy

commutative diagram

$$N_0 \longleftarrow N_1 \longleftarrow N_2 \longleftarrow \cdots$$
$$\overline{N}_0 \longleftarrow \overline{N}_1 \longleftarrow \overline{N}_2 \quad \cdots$$

where each up arrow is a homotopy inverse of the corresponding $N_i \hookrightarrow \overline{N}_i$, shows that the lower sequence defines $\mathcal{S}h(\varepsilon(Y))$. Since $\cap \overline{N}_i = Z$, it also defines the shape of Z. $\square$

Corollary 8.5 (Uniqueness of $\mathcal{Z}$-boundaries up to shape). *All $\mathcal{Z}$-boundaries of a given space Y are shape equivalent. Even more, if Y and Y' are $\mathcal{Z}$-compactifiable and proper homotopy equivalent at infinity, then each $\mathcal{Z}$-boundary of Y is shape equivalent to each $\mathcal{Z}$-boundary of Y'.*

Example 35. *We can now verify the comment at the end of Example 31. For any $\mathcal{Z}$-boundary Z of a Davis manifold $\mathcal{D}^n$, pro-$\pi_1(Z)$ must match the nonstable pro-$\pi_1(\varepsilon(\mathcal{D}^n))$ established in §2.2. Hence Z cannot be an ANR.*

Next we examine the existence question for $\mathcal{Z}$-compactifications. By above results we know that, for reasonably nice X, absolute inward tameness is necessary; moreover, prospective $\mathcal{Z}$-boundaries must come from $\mathcal{S}h(\varepsilon(X))$. It turns out that this is not enough. The outstanding result on this topic, due to Chapman and Siebenmann [ChSi76], provides a complete characterization of $\mathcal{Z}$-compactifiable Hilbert cube manifolds and a model for more general characterizations.

Theorem 8.6 (The Chapman-Siebenmann $\mathcal{Z}$-compactification Theorem). *A Hilbert cube manifold X admits a $\mathcal{Z}$-compactification if and only if each of the following is satisfied.*

(1) *X is inward tame at infinity,*
(2) *$\sigma_\infty(X) \in \varprojlim \left\{ \widetilde{K}_0 \mathbb{Z}[\pi_1(N)] \mid N \text{ a clean nbd. of infinity} \right\}$ is trivial, and*
(3) *$\tau_\infty(X) \in \varprojlim^1 \{ Wh(\pi_1(N)) \mid N \text{ a clean nbd. of infinity} \}$ is trivial.*

(We met Condition (2) in §5.3; (3) will be explained in the proof sketched below.)

Remark 15. (a) Since every ANR Y becomes a Hilbert cube manifold upon crossing with Q, Conditions (1)-(3) are also necessary for Z-compactifiability of an ANR (Although Condition (2) and particularly (3) are best measured in $Y \times Q$.) For some time, it was hoped that those would also be sufficient for ANRs; but in [Gui01], a 2-dimensional polyhedral counterexample was constructed.

(b) If X is 1-ended and pro-$\pi_1(\varepsilon(X))$ is stable, Condition (2) reduces to the simpler Condition (3) in Siebenmann's thesis and Condition (3) of Theorem 8.6 is automatically satisfied. Hence, the hypotheses of Theorem 8.6 are directly comparable to, and significantly weaker than, those of Theorem 5.10.

(c) For those who prefer finite-dimensional spaces, Ferry [Fer00] has shown that, if a k-dimensional locally finite polyhedron P satisfies (1)-(3), then $P \times \mathbb{I}^{2k+5}$ is $\mathcal{Z}$-compactifiable. May [May07] showed that, for the counterexample P_0 from [Gui01], $P_0 \times \mathbb{I}$ is $\mathcal{Z}$-compactifiable. In still-to-be-published work, Guilbault has shown that, for an open manifold M^n satisfying (1)-(3), $M^n \times \mathbb{I}$ is $\mathcal{Z}$-compactifiable.

The following are significant and still open.

Problem 2. *Find conditions that must be added to those of Theorem 8.6 to obtain a characterization of $\mathcal{Z}$-compactifiability for all ANRs.*

Problem 3. *Determine whether the conditions of Theorem 8.6 are sufficient in the case of finite-dimensional manifolds.*

Before describing the proof of Theorem 8.6, we make some obvious adaptations of terminology from §5.2 and 5.3. A clean neighborhood of infinity N in a Hilbert cube manifold X is an *open collar* if $N \approx \mathrm{Bd}_X N \times [0,1)$ and a *homotopy collar* if $\mathrm{Bd}_X N \hookrightarrow N$ is a homotopy equivalence. X is *collarable* if it contains an open collar neighborhood of infinity and *pseudo-collarable* if it contains arbitrarily small homotopy collar neighborhoods of infinity.

Sketch of the proof of Theorem 8.6. The necessity of Conditions (1) and (2) follows from Proposition 8.3; for the necessity of (3), the reader is referred to [ChSi76]. We focus attention here on sufficiency of these conditions.

Assume that X satisfies Conditions (1)-(3). We show that X is $\mathcal{Z}$-compactifiable by showing that it is homeomorphic at infinity to $\mathrm{Tel}\,(\{K_i, f_i\}) \times \mathcal{Q}$, where $\{K_i, f_i\}$ is a carefully chosen inverse sequence of finite polyhedra. Since inverse mapping telescopes are $\mathcal{Z}$-compactifiable (Example 34), the result follows.

It is easiest to read the following argument under the added assumption that X is 1-ended. In the final step, we explain how that assumption can be eliminated.

Step 1. (Existence a pseudo-collar structure) Choose a nested cofinal sequence $\{N_i'\}$ of clean neighborhoods of infinity. By Condition (1) each is finitely dominated, so we have a sequence of Wall finiteness obstructions $(\sigma_0, \sigma_1, \sigma_2, \cdots)$ which determine $\sigma_\infty(X)$; and by (2) each $\sigma_i = 0$, so each N_i' has finite type. (Said differently, Conditions (1) and (2) are equivalent to absolute inward tameness.) For each i, choose a finite polyhedron K_i and an embedding $K_i \hookrightarrow N_i'$ that is a homotopy equivalences. By taking neighborhoods A_i of the K_i, we arrive at a sequence of Hilbert cube manifold pairs (N_i', A_i), where each inclusion is a homotopy equivalence. By some Hilbert cube manifold magic it can be arranged that A_i is a $\mathcal{Z}$-set in N_i' and $\mathrm{Bd}\, N_i' \subseteq A_i$. From there one finds $N_i \subseteq N_i'$ for which $\mathrm{Bd}\, N_i$ is a copy of A_i and $\mathrm{Bd}\, N_i \hookrightarrow N_i$ a homotopy equivalence (see [ChSi76] for details). Thus $\{N_i\}$ is a pseudocollar structure.

Step 2. (Pushing the torsion off the end of X) By letting $A_i = \overline{N_i - N_{i+1}}$ for each i, view the end of X as a countable union $A_0 \cup A_1 \cup A_2 \cup \cdots$ of compact 1-sided h-cobordisms $(A_i, \mathrm{Bd}\, N_i, \mathrm{Bd}\, N_{i+1})$ of Hilbert cube manifolds. (See Exercise 5.11.) By the triangulability of Hilbert cube manifolds (and pairs), each inclusion $\mathrm{Bd}\, N_i \hookrightarrow A_i$ has a well-defined torsion $\tau_i \in \mathrm{Wh}\,(\pi_1(N_i))$. Together these torsions determine a

representative $(\tau_0, \tau_1, \tau_2, \cdots)$ of $\tau_\infty (X)$. (**Note.** Determining $\tau_\infty (X)$ requires that Step 1 first be accomplished; there is no $\tau_\infty (X)$ without Conditions (1) and (2) being satisfied.)

We would like to alter the choices of the N_i by using an infinite borrowing strategy like that employed in the proof of Theorem 5.10. In particular, we would like to borrow a compact Hilbert cube manifold h-cobordism B_0 from a collar neighborhood of $\underline{\mathrm{Bd}\, N_1}$ in A_1 so that $\mathrm{Bd}\, N_0 \hookrightarrow A_0 \cup B_0$ has trivial torsion. Then, replacing N_1 with $\overline{N_1 - B_0}$, we would like to borrow B_1 from A_2 so that so that $\mathrm{Bd}\, N_1 \hookrightarrow A_1 \cup B_1$ has trivial torsion. Continuing inductively, we would like to arrive at an adjusted sequence $N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots$ of neighborhoods of infinity for which each $\mathrm{Bd}\, N_i \hookrightarrow A_i$ has trivial torsion.

The "derived limit", $\varprojlim^1$, is defined precisely to measure whether this infinite borrowing strategy can be successfully completed. In the situation of 5.10, where the fundamental group stays constant from one side of each A_i to the other, there was no obstruction to the borrowing scheme. More generally, as long as the inclusion induced homomorphisms $\mathrm{Wh}\,(\pi_1\,(N_{i+1})) \to \mathrm{Wh}\,(\pi_1\,(N_i))$ are surjective for all i, the strategy works. But, in general, we must rely on the hypothesis that $\tau_\infty (X)$ is the trivial element of $\varprojlim^1 \{Wh(\pi_1\,(N)) \mid N$ a homotopy collar nbd. of infinity$\}$.

Step 3. (Completion of the proof) Homotopy equivalences $K_i \hookrightarrow \mathrm{Bd}\, N_i$ and the deformation retractions of A_i onto $\mathrm{Bd}\, N_i$ determine maps $f_{i+1} : K_{i+1} \to K_i$ and homotopy equivalences of triples $(A_i, \mathrm{Bd}\, N_i, \mathrm{Bd}\, N_{i+1}) \simeq (\mathrm{Map}\,(f_{i+1}), K_i, K_{i+1})$. Using the fact that both $\mathrm{Bd}\, N_i \hookrightarrow A_i$ and $K_i \hookrightarrow \mathrm{Map}\,(f_{i+1})$ have trivial torsion (and through more Hilbert cube manifold magic), we obtain a homeomorphism of triples $(A_i, \mathrm{Bd}\, N_i, \mathrm{Bd}\, N_{i+1}) \simeq (\mathrm{Map}\,(f_{i+1}) \times \mathcal{Q}, K_i \times \mathcal{Q}, K_{i+1} \times \mathcal{Q})$. Piecing these together gives a homeomorphism $N_0 \approx \mathrm{Tel}\,(\{K_i, f_i\}) \times \mathcal{Q}$, and completes the proof.

As a mild alternative, we could have used the ingredients described above to construct a proper homotopy equivalence $h : N_0 \to \mathrm{Tel}\,(K_i, f_i)$ and used the triviality of the torsions to argue that h is an infinite simple homotopy equivalence, in the sense of [Sie70]. Then, by a variation on Theorem B.6, $\widehat{h} : N_0 \to \mathrm{Tel}\,(K_i, f_i) \times \mathcal{Q}$ is homotopic to a homeomorphism.

Final Step. (Multi-ended spaces) When X is multi-ended, the neighborhoods of infinity N used to define σ_∞ and τ_∞ can have multiple (but finitely many) components. In those cases, define $\widetilde{K}_0 \mathbb{Z}\,[\pi_1\,(N)]$ and $Wh(\pi_1\,(N))$ to be the direct sums $\bigoplus \widetilde{K}_0 \mathbb{Z}\,[\pi_1\,(C_j)]$ and $\bigoplus Wh(\pi_1\,(C_j))$, where $\{C_j\}$ is the collection of components of N. The fact that reduced projective class groups and the Whitehead groups of free product are the corresponding direct sums allows the above steps to go through as stated. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 16. If desired, one can arrange, in the final lines of Step 3, a homeomorphism $k : X \to \mathrm{Tel}^*\,(K_i, f_i) \times \mathcal{Q}$, defined on all of X. The space on the right is the previous mapping telescope with the addition of a single mapping cylinder $\mathrm{Map}(K_0 \xrightarrow{f_0} K_{-1})$. The finite complex K_{-1} and the map f_0 are carefully chosen so that X and $\mathrm{Tel}^*\,(K_i, f_i)$ are infinite simple homotopy equivalent.

Step 1 of the above proof provides a result that is interesting in its own right.

Theorem 8.7. *A Hilbert cube manifold is pseudo-collarable if and only if it satisfies Conditions (1) and (2) of Theorem 8.6 or, equivalently, if and only if it is absolutely inward tame.*

It is interesting to compare 8.7 to Theorems 5.17 and 5.18.

9. $\mathcal{Z}$-BOUNDARIES IN GEOMETRIC GROUP THEORY

In this section we look at the role of $\mathcal{Z}$-compactifications and $\mathcal{Z}$-boundaries in geometric group theory.

9.1. **Boundaries of δ-hyperbolic groups.** For every δ-hyperbolic metric space (X, d), Gromov [Gro87] defined a *boundary*, ∂X, and a corresponding compactification $\widehat{X} = X \cup \partial X$. This boundary is not only well-defined, it is a quasi-isometry invariant in the following strong sense: if $f : X \to Y$ is a quasi-isometry between δ-hyperbolic spaces, then there is a unique extension $\widehat{f} : \widehat{X} \to \widehat{Y}$ that that restricts to a homeomorphism between boundaries. This is of particular interest when G is a finitely generated group endowed with a corresponding word metric. It is a standard fact that, for any two such metrics, $G \overset{\mathrm{id}}{\to} G$ is a quasi-isometry; so for a δ-hyperbolic group G, the *Gromov boundary* ∂G is well-defined.

Early in the study of δ-hyperbolic groups, it became clear that exotic topological spaces arise naturally as group boundaries. In addition to spheres of all dimensions, the collection of known boundaries includes: Cantor sets, Sierpinski carpets, Menger curves, Pontryagin surfaces, and 2-dimensional Menger spaces, to name a few. See [Bes96], [Dra99] and [KaBe02]. So it is not surprising that shape theory has a role to play in this area. But, a priori, Gromov's compactifications and boundaries have little in common with $\mathcal{Z}$-compactifications and $\mathcal{Z}$-boundaries. After all, for a word hyperbolic group, the Gromov compactification adds boundary to a *discrete* topological space.

Exercise 9.1. *Show that a countably infinite discrete metric space does not admit a $\mathcal{Z}$-compactification.*

In 1991, Bestvina and Mess introduced the use of $\mathcal{Z}$-boundaries and $\mathcal{Z}$-compactifications to the study of δ-hyperbolic groups. For a discrete metric space (X, d) and a constant ρ, the *Rips complex* $P_\rho(G)$ is the simplicial complex obtained by declaring the vertex set to be X and filling in an n-simplex for each collection $\{x_0, x_1, \cdots, x_n\}$ with $d(x_i, x_j) \leq \rho$ for all $0 \leq i, j \leq n$. Clearly, a Rips complex $P_\rho(G)$ for a finitely generated group G admits a proper, cocompact G-action and $G \hookrightarrow P_\rho(G)$ is a quasi-isometry. So when G is δ-hyperbolic there is a canonical compactification $\overline{P_\rho(G)} = P_\rho(G) \cup \partial G$. Furthermore, it was shown by Rips that, for δ-hyperbolic G and large ρ, $P_\rho(G)$ is a contractible. Using this and some finer homotopy properties of $P_\rho(G)$, Bestvina and Mess proved the following.

Theorem 9.2 ([BeMe91, Th.1.2])**.** *Let G be a δ-hyperbolic group and $\rho \geq 4\delta + 2$, then $\overline{P_\rho(G)} = P_\rho(G) \cup \partial G$ is a $\mathcal{Z}$-compactification.*

Implications of Theorem 9.2 are cleanest when $P_\rho(G)$ is a cocompact EG complex. Since contractibility and a proper cocompact action have already been established, only freeness is needed, and that is satisfied if and only if G is torsion-free.

Corollary 9.3. *Let G be a torsion-free word hyperbolic group. Then*

(1) *every cocompact EG complex is inward tame and proper homotopy equivalent to $P_\rho(G)$,*

(2) *for every cocompact EG complex X, $\mathcal{S}h(\varepsilon(X)) = \mathcal{S}h(\partial G)$,*

(3) *pro-$H_*(G; R)$, $\check{H}_*(G; R)$ and $\check{H}^*(G; R)$ are isomorphic to the corresponding invariants of ∂G,*

(4) *for 1-ended G, pro-$\pi_*(G)$ and $\check{\pi}_*(G)$ are well defined and isomorphic to the corresponding invariants of $\mathcal{S}h(\partial G)$,*

(5) $H^*(G; RG) \cong \check{H}^{*-1}(\partial G; R)$ *for any coefficient ring R.*

Proof of Corollary. The discussion in §6.1 explains why all cocompact EG complexes are proper homotopy equivalent. Since one such space, $P_\rho(G)$, is $\mathcal{Z}$-compactifiable and therefore inward tame, they are all inward tame. By Theorem 8.4, $\mathcal{S}h(\varepsilon(P_\rho(G)))$ $= \mathcal{S}h(\partial G)$, so Theorem 7.14 completes (2). Assertion (3) is a consequence of Theorem 7.13, while 4) is similar, except that Theorem 6.5 is used to assure well-definedness. Assertion 5), a statement about group cohomology with coefficients in RG, requires some algebraic topology that is explained in [BeMe91]; it is essentially a consequence of (3). □

For the most part, Corollary 9.3 is all about the shape of ∂G and the relationship between a $\mathcal{Z}$-boundary and its complement. There are other applications of boundaries of δ-hyperbolic groups that use more specific properties of ∂G. Here is a small sampling:

- [BeMe91] provides formulas relating the cohomological dimension of a torsion-free G to the topological dimension of ∂G. (Clearly, the latter is not a shape invariant.)
- The semistability of G was deduced by proving that ∂G has no cut points [Swa96], and therefore is locally connected, by results from [BeMe91].
- By work from [Tuk88],[Gab92],[CaJu94] and [Fre95], $\partial G \approx \mathbb{S}^1$ if and only if G is virtually the fundamental group of a closed hyperbolic surface.
- Bowditch [Bow98] has obtained a JSJ-decomposition theorem for δ-hyperbolic groups by analyzing cut pairs in ∂G.
- See [KaBe02] for many more examples.

9.2. **Boundaries of CAT(0) groups.** Another widely studied class of groups are the *CAT(0) groups*, ie, groups G that act geometrically (properly and cocompactly by isometries) on a proper CAT(0) space. If X is such a CAT(0) space, the visual boundary $\partial_\infty X$ is called a *group boundary* for G. Since a given G may act geometrically on multiple proper CAT(0) spaces, it is not immediate that its boundary is topologically well-defined; in fact it is not. The first example was discovered by Croke and Kleiner [CrKl00]. Their work was expanded upon by Wilson [Wil05], who showed that the

same group admits a continuum of topologically distinct boundaries. Mooney [Moo08] discovered additional examples from the category of knot groups, and in [Moo10] produced examples with boundaries any dimension $k \geq 1$. This situation suggests that CAT(0) boundaries (being ill-defined) are not likely to be useful—but that is not the case. One reason is the following "approximate well-definedness" result.

Theorem 9.4 (Uniqueness of CAT(0) boundaries up to shape). *All CAT(0) boundaries of a given CAT(0) group G are shape equivalent.*

Proof. If G is torsion-free, then a geometric G-actions on a proper CAT(0) space X is necessarily free so, X is an EG space. It follows that all CAT(0) spaces on which G acts geometrically are proper homotopy equivalent. So, by Corollary 8.5, all CAT(0) boundaries of G have the same shape.

If G has torsion there is more work to be done, but the idea is the same. In [Ont05], Ontaneda shows that any two proper CAT(0) spaces on which G acts geometrically are proper homotopy equivalent, so again their boundaries have the same shape. $\square$

Aa an application of Theorem 9.4, Corollary 9.3 can be repeated for torsion-free CAT(0) groups, with two exceptions: **(a)** we must omit reference to the Rips complex since it is not known to be an EG for CAT(0) groups, and **(b)** in general, pro-$\pi_*(G)$ and $\check{\pi}_*(G)$ are not known to be well-defined since the following is open.

Conjecture 6 (CAT(0) Semistability Conjecture). *Every 1-ended CAT(0) group G is semistable.*

It is worth noting that ∂G, itself, provides an approach to this conjecture. By a result from shape theory [DySe78, Th.7.2.3], G is semistable if and only if ∂G has the shape of a locally connected compactum. (This is true in much greater generality.)

Before moving away from CAT(0) group boundaries, we mention a few more applications.

- The Bestvina-Mess formulas, mentioned earlier, relating the cohomological dimension of a torsion-free G to the topological dimension of ∂G are also valid for CAT(0) G.
- Swenson [Swe99] has shown that a CAT(0) group G with a cut point in ∂G has an infinite torsion subgroup.
- Ruane has shown that for CAT(0) G, if ∂G is a circle, then G is virtually $\mathbb{Z} \times \mathbb{Z}$ or the fundamental group of a closed hyperbolic surface; and if ∂G is a suspended Cantor set, then G is virtually $\mathbb{F}_2 \times \mathbb{Z}$.
- Swenson and Papasoglu [PaSw09] have, in a manner similar to Bowditch's work on δ-hyperbolic groups, used cut pairs in ∂G to obtain a JSJ-decomposition result for CAT(0) groups.

9.3. **A general theory of group boundaries.** Motivated by their usefulness in the study of δ-hyperbolic and CAT(0) groups, Bestvina [Bes96] developed an axiomatic approach to group boundaries which unified the existing theories and provided a framework for defining group boundaries more generally. We begin with the original definition, then introduce some variations.

A *$\mathcal{Z}$-structure on a group G* is a topological pair $\left(\overline{X}, Z\right)$ satisfying the following four conditions:

(1) $\overline{X}$ is a compact ER,
(2) Z is a $\mathcal{Z}$-set in $\overline{X}$,
(3) $X = \overline{X} - Z$ admits a proper, free, cocompact G-action, and
(4) the G-action satisfies the following *nullity condition*: for every compactum $A \subseteq X$ and every open cover $\mathcal{U}$ of $\overline{X}$, all but finitely many G-translates of A are $\mathcal{U}$-small, ie, are contained in some element of $\mathcal{U}$.

A pair $\left(\overline{X}, Z\right)$ that satisfies (1)-(3), but not necessarily (4) is called a *weak $\mathcal{Z}$-structure* on G, while a $\mathcal{Z}$-structure on G that satisfies the additional condition:

(5) The G-action on X extends to a G-action on $\overline{X}$,

is called an *$\mathcal{EZ}$-structure* (or an *equivariant $\mathcal{Z}$-structure*) on G. A *weak $\mathcal{EZ}$-structure* is a weak $\mathcal{Z}$-structure that satisfies Condition (5)[10].

Under the above circumstances, Z is called a *$\mathcal{Z}$-boundary*, a *weak $\mathcal{Z}$-boundary*, an *$\mathcal{EZ}$-boundary*, or a *weak $\mathcal{EZ}$-boundary*, as appropriate.

Example 36 (A sampling of $\mathcal{Z}$-structures).

(1) *The $\mathcal{Z}$-compactification $\overline{P_\rho(G)} = P_\rho(G) \cup \partial G$ of Theorem 9.2 is an $\mathcal{EZ}$-structure whenever G is a torsion-free δ-hyperbolic group.*

(2) *If a torsion-free group G admits a geometric action on a finite-dimensional proper CAT(0) space X, then $\overline{X} = X \cup \partial_\infty X$ is an $\mathcal{EZ}$-structure for G.*

(3) *The Baumslag-Solitar group $BS(1,2) = \langle a, b \mid bab^{-1} = a^2 \rangle$ is put forth by Bestvina as an example that is neither δ-hyperbolic nor CAT(0), but still admits a $\mathcal{Z}$-structure. The $\mathcal{Z}$-structure described in [Bes96] is also an $\mathcal{EZ}$-structure. The traditional cocompact EG 2-complex for $BS(1,2)$ is homeomorphic to $\mathbb{T}_3 \times \mathbb{R}$, where $\mathbb{T}_3$ is the uniformly trivalent tree. Given the Euclidean product metric, $\mathbb{T}_3 \times \mathbb{R}$ is CAT(0), so adding the visual boundary gives a weak $\mathcal{Z}$-structure, with a suspended Cantor set as boundary. (Since the action of $BS(1,2)$ on $\mathbb{T}_3 \times \mathbb{R}$ is not by isometries, one cannot conclude that $BS(1,2)$ is CAT(0)). This weak $\mathcal{Z}$-structure does not satisfy the nullity condition— instead it provides a nice illustration of the failure of that condition. Nevertheless, by using this structure as a starting point, a genuine $\mathcal{Z}$-structure (in fact more than one) can be obtained.*

(4) *Januszkiewicz and Świątkowski [JaSw06] have developed a theory of "systolic" spaces and groups that parallels, but is distinct from, CAT(0) spaces and groups. Among systolic groups are many that are neither δ-hyperbolic nor*

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[10]Bestvina infomally introduced the definition of weak $\mathcal{Z}$-structure in [Bes96], where he also commented on his decision to omit Condition 5) from the definition of $\mathcal{Z}$-structure. Farrell and Lafont introduced the term $\mathcal{EZ}$-structure in [FaLa05].

CAT(0). A delicate construction by Osajda and Przytycki in [OsPr09] places $\mathcal{EZ}$-structures on all torsion-free systolic groups.

(5) *Dahmani [Dah03] showed that, if a group G is hyperbolic relative to a collection of subgroups, each of which admits a $\mathcal{Z}$-structure, then G admits a $\mathcal{Z}$-structure.*

(6) *Tirel [Tir11] showed that if G and H each admit $\mathcal{Z}$-structures (resp., $\mathcal{EZ}$-structures), then so do $G \times H$ and $G * H$.*

Exercise 9.5. *Verify the assertion made in Item (2) of Example 36.*

Exercise 9.6. *For $G \times H$ in Item 6), give an easy proof of the existence of weak $\mathcal{Z}$-structures (resp., weak $\mathcal{EZ}$-structures). As with Item (3) the sticking point is the nullity condition.*

Given the wealth of examples, it becomes natural to ask whether all reasonably nice groups admits $\mathcal{Z}$-structures. The following helps define "reasonably nice".

Proposition 9.7. *A group G that admits a weak $\mathcal{Z}$-structure must have type F.*

Proof. If $(\overline{X}, Z)$ is a weak $\mathcal{Z}$-structure on G, then $X = \overline{X} - Z$ is an EG and $X \to G\backslash X$ is a covering projection. Since being an ENR is a local property, $G\backslash X$ is an ENR; it is also compact and aspherical. By Proposition A.2, $G\backslash X$ is homotopy equivalent to a finite complex K, which is a $K(G, 1)$. $\square$

Question 2 (all are open)**.** *Does every group of type F admit a $\mathcal{Z}$-structure? an $\mathcal{EZ}$-structure? a weak $\mathcal{Z}$-structure? a weak $\mathcal{EZ}$-structure?*

The first of the above questions was asked explicitly by Bestvina in [Bes96], where he also mentions the version for weak $\mathcal{Z}$-structures. The latter of those two was also mentioned in [BeMe91], where the weak $\mathcal{EZ}$-version is explicitly asked. The $\mathcal{EZ}$-version, was suggested by Farrell and Lafont in [FaLa05].

In [Bes96], Bestvina prefaced his posing of the $\mathcal{Z}$-structure Question with the warning: *"There seems to be no systematic method of constructing boundaries of groups in general, so the following is probably hopeless."* In the years since that question was posed, a general has not yet emerged. However, there have been successes (36) when attention is focused on a specific group or classes of groups. In private conversations and in presentations, Bestivina has suggested some additional groups for consideration; among them, the groups $Out(F_n)$ and the various Baumslag-Solitar groups $BS(m, n)$. Farrell and Lafont have specifically asked about $\mathcal{EZ}$-structures for torsion-free finite index subgroups of $SL_n(\mathbb{Z})$. A less explicit, but highly important class of groups, are the fundamental groups of closed aspherical manifolds (or more generally, Poincaré duality groups)—the hope being that well-developed tools from manifold topology might provide an advantage.

Bestvina [Bes96, Lemma 1.4] has shown that if G admits a $\mathcal{Z}$-structure $(\overline{X}, Z)$, then every cocompact EG complex Y can be incorporated into a $\mathcal{Z}$-structure $(\overline{Y}, Z)$. In particular, every cocompact EG complex the hypotheses of Theorem 8.6. So it seems natural to begin with:

Question 3. *Must the universal cover of a finite aspherical complex be inward tame? absolutely inward tame?*

Remarkably, nothing seems to be known here. An early version of the question goes back to [Geo86], with more explicit formulations found in [Gui00] and [Fer00].

Since, for fixed G, all cocompact EG spaces are proper homotopy equivalent, we can view *inward tameness* as a property possessed by some (possibly all) type F groups. Moreover, if G is inward tame, we can use §7.4 to define the *shape of the end of G*. Specifically, for X a cocompact EG, $\mathcal{S}h(\varepsilon(G)) = \mathcal{S}h(\varepsilon(X))$. If $K \in \mathcal{S}h(\varepsilon(G))$, we might even view K as a "pre-$\mathcal{Z}$-boundary" and (X, K) as a "pre-$\mathcal{Z}$-structure" for G.

As for applications of the various sorts of $\mathcal{Z}$-boundaries, we list a few.

- As noted in the previous paragraph, even pre-$\mathcal{Z}$-boundaries are well-defined up to shape. So a result like 9.3 can be stated here, with the same exceptions as noted above for CAT(0) groups.
- In [Bes96], it is shown that the Bestvina-Mess formulas relating the cohomological dimension of a torsion-free G to the topological dimension of ∂G are again valid for $\mathcal{Z}$-boundaries. For this, the full strength of Bestvina's definition of $\mathcal{Z}$-structure is used.
- Carlsson and Pedersen [CaPe95] and Farrell and Lafont [FaLa05] have shown that groups admitting an $\mathcal{EZ}$-structure satisfy the Novikov Conjecture.

9.4. **Further generalizations.** A pair of generalizations to the $(\mathcal{E})\mathcal{Z}$-structure/ boundary definitions can be found in the literature[11].

i) Replace the requirement that $\overline{X}$ be an ER with the weaker requirement that it be an AR.

ii) Drop the freeness requirement for the G-action on X.

Change i) simply allows $\overline{X}$ to be infinite-dimensional; by itself that may be of little consequence. After all, X is still a cocompact EG, so there exists a finite $K(G, 1)$ complex K. If Z is finite-dimensional, Bestvina's boundary swapping trick ([Bes96, Lemma 1.4]) produces a new $\mathcal{Z}$-structure $(\overline{Y}, Z)$ in which $\overline{Y}$ is an ER. This motivates the question:

Question 4. *If $(\overline{X}, Z)$ is a $\mathcal{Z}$-structure on a group G in the sense of [Bes96], except that $\overline{X}$ is only required to be an ANR, must Z still be finite-dimensional? (Compare to [Swe99, Th.12], which shows that a CAT(0) group boundary is finite-dimensional, regardless of the CAT(0) space it bounds.)*

Change ii) is more significant; it allows for groups with torsion. $\mathcal{Z}$-structures of this sort are plentiful in the categories of δ-hyperbolic and CAT(0) groups, with Coxeter groups the prototypical examples, so this generalization very natural. There are, however, complications. When G has torsion, the notion of a cocompact EG complex

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[11]For example, both are incorporated into [Dra06].

must be replaced by that of a cocompact (or G-finite) $\underline{E}G$ complex, where G may act with fixed points, subject to the requirement that stabilizers of all finite subgroups are contractible subcomplexes. This notion is fruitful and cocompact $\underline{E}G$ complexes, when they exist, are well-defined up to G-equivariant homotopy equivalence, and more importantly (from the point of view of these notes) up to proper homotopy equivalence.

In order to obtain the sorts of conclusions we are concerned with here, positive answers to the following, questions would be of interest.

Question 5. *Suppose G admits a $\mathcal{Z}$-structure $\left(\overline{X}, Z\right)$, but with the G-action on X not required to be free. If $\left(\overline{X'}, Z'\right)$ is another such $\mathcal{Z}$-structure, is $X \overset{p}{\simeq} X'$? More specifically, does there exist a cocompact $\underline{E}G$ complex and must X be proper homotopy equivalent to that complex?*

10. $\mathcal{Z}$-boundaries in manifold topology

In this section we look specifically at $\mathcal{Z}$-compactifications and $\mathcal{Z}$-boundaries of manifolds, with an emphasis on open manifolds and manifolds with compact boundary. In §9 we noted the occurrence of many exotic group boundaries: Cantor sets, suspended Cantor sets, Hawaiian earrings, Sierpinski carpets, and Pontryagin surfaces, to name a few. By contrast, we will see that a $\mathcal{Z}$-boundary of an n-manifold with compact boundary is always a homology $(n-1)$-manifold. That does not mean the $\mathcal{Z}$-boundary is always nice—recall Example 31—but it does mean that manifold topology forces some significant regularity on potential $\mathcal{Z}$-boundaries. Here we take a look at that result and some related applications. First, a quick introduction to homology manifolds.

10.1. **Homology manifolds.** If N^n is an n-manifold with boundary, then each $x \in$ int N^n has *local homology*

$$\widetilde{H}_* \left(N^n, N^n - x\right) \cong \widetilde{H}_* \left(\mathbb{R}^n, \mathbb{R}^n - \mathbf{0}\right) \cong \left\{ \begin{array}{ll} \mathbb{Z} & \text{if } * = n \\ 0 & \text{otherwise} \end{array} \right.$$

and each $x \in \partial N^n$ has local homology $\widetilde{H}_* \left(N^n, N^n - x\right) \cong \widetilde{H}_* \left(\mathbb{R}^n_+, \mathbb{R}^n_+ - \mathbf{0}\right) \equiv 0$. This motivates the notion of a "homology manifold".

Roughly speaking, X is a *homology n-manifold* if $\widetilde{H}_* \left(X, X - x\right) \cong \widetilde{H}_* \left(\mathbb{R}^n, \mathbb{R}^n - \mathbf{0}\right)$ for all $x \in X$, and a *homology n-manifold with boundary* if

$$\widetilde{H}_* \left(X, X - x\right) \cong \widetilde{H}_* \left(\mathbb{R}^n, \mathbb{R}^n - \mathbf{0}\right) \quad \text{or} \quad \widetilde{H}_* \left(X, X - x\right) \cong \widetilde{H}_* \left(\mathbb{R}^n_+, \mathbb{R}^n_+ - \mathbf{0}\right) \equiv 0$$

for all $x \in X$. In the latter case we define

$$\partial X \doteq \left\{ x \in X \mid \widetilde{H}_* \left(X, X - x\right) = 0 \right\}$$

and call this set the *boundary of X*.[12]

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[12]All homology here is with $\mathbb{Z}$-coefficients. With the same strategy and an arbitrary coefficient ring, we could also define *R-homology manifold.*

The reason for the phrase "roughly speaking" in the above paragraph is because ordinary singular homology theory does not always detect the behavior we are looking for. This issue is analogous to what happened in shape theory: there, when singular theory told us that the homology of the Warsaw circle W was the same as that of a point, we developed Čech homology theory to better capture the circle-like nature of W. In the current setting, we again need to adjust our homology theory to match our goals. Without going into detail, we simply state that, for current purposes, Borel-Moore homology theory should be used. Moreover, since Borel-Moore homology of a pair requires that A be closed in X, we interpret $\widetilde{H}_*(X, X - x)$ to mean $\varinjlim \widetilde{H}_*(X, X - U)$ where U varies over all open neighborhoods of x.

With the above adjustment in place, we are nearly ready to discuss the essentials of homology manifolds. Before doing so we note that there is an entirely analogous theory of *cohomology manifolds*, in which Alexander-Čech theory is the preferred cohomology theory. We also note that both Borel-Moore homology and Alexander-Čech cohomology theories agree with the singular theories when X is an ANR. An ANR homology manifold is often called a *generalized manifold*—a class of objects that plays an essential role in geometric topology.

Example 37. *Let Σ^n be a non-simply connected n-manifold with the same $\mathbb{Z}$-homology as $\mathbb{S}^n$, e.g., the boundary of a Newman contractible $(n+1)$-manifold. Then $X = \operatorname{cone}(\Sigma^n) = \Sigma^n \times [0,1] / \{\Sigma^n \times 1\}$ is a homology $(n+1)$-manifold with boundary, where $\partial X = \Sigma^n \times 0$. The double of X^{n+1}, the suspension of Σ^{n+1}, is a homology manifold that is homotopy equivalent to $\mathbb{S}^{n+1}$. Both of these are ANRs, hence generalized manifolds, but neither is an actual manifold.*

Example 38. *Let A be a compact proper subset of an n-manifold M^n and let $Y = M^n/A$ be the quotient space obtained by identifying A to a point. If A is cell-like (ie, has trivial shape), then X is a generalized n-manifold. In many cases $Y \approx M^n$, for example, when A is a tame arc or ball. In other cases—for example, A a wild arc with non-simply connected complement or A a Newman contractible n-manifold embedded in M^n, Y is not a manifold.*

Remark 17. The subject of *Decomposition Theory* is motivated by Example 38. There, the following question is paramount: Given a pairwise disjoint collection G of cell-like compacta in a manifold M^n satisfying a certain niceness condition (an *upper semicontinuous decomposition*), when is the quotient space $M/\mathcal{G}$ a manifold? Although the premise sounds simple and very specific, results from this area have had broad-ranging impacts on geometric topology, including: existence of exotic involutions on spheres, existence of exotic manifold factors (non-manifolds X for which $X \times \mathbb{R}$ is a manifold), existence of non-PL triangulations of manifolds, and a solution to the 4-dimensional Poincaré Conjecture. The Edwards-Quinn Manifold Recognition Theorem, which will be used shortly, belongs to this subject. References to the "Moore-Bing school of topology" usually indicate work by R.L. Moore, R H Bing, and their mathematical descendents in this area. See [Dvr86] for a comprehensive discussion.

Exercise 10.1. *Here we describe a simple non-ANR homology manifold. Let H^n be non-simply connected n-manifold with the homology of a point and a boundary homeomorphic to $\mathbb{S}^{n-1}$, and let $\{B_i^n\}_{i=1}^\infty$ be a sequence of pairwise disjoint round n-balls in $\mathbb{S}^n$ converging to a point p. Create X by removing the interiors the B_i^n and replacing each with $H_i^n \approx H^n$. Topologize X so that $\{H_i^n\}$ is a null sequence in X. The result is a homology manifold (some knowledge of Borel-Moore homology is needed to verify this fact).*

Explain why X is not an ANR. Then show that X does not satisfy the definition of homology manifold if singular homology is used.

Exercise 10.2. *Show that the $\mathcal{Z}$-boundary attached to the Davis manifold described in Example 31 is homeomorphic to the non-ANR homology manifold described in Exercise 10.1 (some attention must be paid to orientations).*

For now, the reader may wish to treat the following theorem as an axiom; [AnGu99] shows how the classical literature can be woven together for a proof.

Theorem 10.3 (Fundamental facts about (co)homology manifolds)**.**

(1) *A space X is a homology n-manifold if and only if it is a cohomology n-manifold.*
(2) *The boundary of a (co)homology n-manifold is a (co)homology $(n-1)$-manifold without boundary.*
(3) *The union of two (co)homology n-manifolds along a common boundary is a (co)homology n-manifold.*
(4) *(Co)Homology manifolds are locally path connected.*

Corollary 10.4. *Let M^n be an open n-manifold (or even just an open generalized manifold) and $\overline{M^n} = M^n \cup Z$ be a $\mathcal{Z}$-compactification. Then*

(1) *$\overline{M^n}$ is a homology n-manifold (with boundary),*
(2) *$\partial \overline{M^n} = Z$, and*
(3) *Z is a homology $(n-1)$-manifold.*

Proof of Corollary. For (1) and (2) we need only check that $H_*\left(\overline{M^n}, \overline{M^n} - z\right) \equiv 0$ at each $z \in Z$. Since $\overline{M^n}$ is an ANR, we are free to use singular homology in place of Borel-Moore theory. A closed subset of a $\mathcal{Z}$-set is a $\mathcal{Z}$-set, so $\{z\}$ is a $\mathcal{Z}$-set in $\overline{M^n}$, and hence, $\overline{M^n} - z \hookrightarrow \overline{M^n}$ is a homotopy equivalence. The desired result now follows from the long exact sequence for pairs.

Item (3) is now immediate from Theorem 10.3. □

Exercise 10.5. *Show that if M^n is a CAT(0) n-manifold, then every metric sphere in M^n is a homology $(n-1)$-manifold.*

Remark 18. In addition to Theorem 10.3, it is possible to define orientation for homology manifolds and prove a version of Poincaré duality for the orientable ones. With those tools, one can also prove, for example, that any $\mathcal{Z}$-boundary of a contractible open n-manifold has the (global) homology of an $(n-1)$-sphere.

Before moving to applications, we state without proof one of the most significant results in this area. A nice exposition can be found in [Dvr86].

Theorem 10.6 (Edwards-Quinn Manifold Recognition Theorem)**.** *Let X^n ($n \geq 5$) be a generalized homology n-manifold without boundary and suppose X^n contains a nonempty open set $U \approx \mathbb{R}^n$. Then X^n is an n-manifold if and only if it satisfies the disjoint disks property (DDP).*

A space X satisfies the DDP if, for any pair of maps $f, g : D^2 \to X$ and any $\varepsilon > 0$, there exist ε-approximations f' and g' of f and g, to that $f'(D^2) \cap g'(D^2) = \varnothing$.

10.2. **Some applications of $\mathcal{Z}$-boundaries to manifold topology.** Most results in this section come from [AnGu99]. Here we provide only the main ideas; for details, the reader should consult the original paper. For the sake of brevity, we focus on high-dimensional results. In many cases, low-dimensional analogs are true for different reasons.

Let $\overline{M^n} = M^n \cup Z$ be a $\mathcal{Z}$-compactification of an open n-manifold. Since $\overline{M^n}$ need not be a manifold with boundary, the following is a pleasant surprise.

Theorem 10.7. *Suppose $\overline{M^n} = M^n \cup Z$ and $\overline{N^n} = N^n \cup Z'$ are $\mathcal{Z}$-compactification of open n-manifolds ($n > 4$) and $h : Z \to Z'$ is a homeomorphism. Then $P^n = \overline{M^n} \cup_h \overline{N^n}$ is a closed n-manifold.*

Sketch of proof. Theorem 10.3 asserts that P^n is a homology n-manifold. From there one uses delicate properties of homology manifolds to prove that P^n is locally contractible at each point on the "seam", $Z = Z'$; hence, P^n is an ANR. Another delicate, but more straightforward, argument (this part using the fact that Z and Z' are $\mathcal{Z}$-sets) verifies the DDP for P^n. An application of Edwards-Quinn completes the proof. □

Corollary 10.8. *The double of $\overline{M^n}$ along Z is an n-manifold. If M^n is contractible, that double is homeomorphic to $\mathbb{S}^n$, and there is an involution of $\mathbb{S}^n$ with Z as its fixed set.*

Sketch of proof. Double($\overline{M^n}$) $\approx \mathbb{S}^n$ will follow from the Generalized Poincaré conjecture if we can show that it is a simply connected manifold with the homology of an n-sphere. The involution interchanges the two copies of $\overline{M^n}$.

That Double($\overline{M^n}$) has the homology of $\mathbb{S}^n$ is a consequence of Mayer-Vietoris and Remark 18. Since $\overline{M^n}$ is simply connected, simple connectivity of Double($\overline{M^n}$) would follow directly from van Kampen's Theorem if the intersection between the two copies was nice. Instead a controlled van Kampen-like argument is employed. Use the fact that homology manifolds are locally path connected to divide an arbitrary loop into loops lying in one or the other copy of $\overline{M^n}$, where they can be contracted. Careful control is needed, and the fact that $\overline{M^n}$ is locally contractible is important. □

Theorem 10.9. *If contractible open manifolds M^n and N^n admit $\mathcal{Z}$-compactifications with homeomorphic $\mathcal{Z}$-boundaries, then $M^n \approx N^n$.*

Sketch of proof. Let Z denote the common $\mathcal{Z}$-boundary. The argument used in Corollary 10.8 shows that the union of these compactifications along Z is $\mathbb{S}^n$. Let $W^{n+1} = \mathbb{B}^{n+1} - Z$ and note that $\partial W^{n+1} = M^n \sqcup N^n$, providing a noncompact cobordism (W^{n+1}, M^n, N^n). The proof is completed by applying the Proper s-cobordism Theorem [Sie70] to conclude that $W^{n+1} \approx M^n \times [0,1]$. That requires some work. First show that $M^n \hookrightarrow W^{n+1}$ is a proper homotopy equivalence. (The fact that Z is a $\mathcal{Z}$-set in $\mathbb{B}^{n+1}$ is key.) Then, to establish that $M^n \hookrightarrow W^{n+1}$ is an infinite simple homotopy equivalence, some algebraic obstructions must be checked. Fortunately, there are "naturality results" from [ChSi76] that relate those obstructions to the $\mathcal{Z}$-compactifiability obstructions for M^n and W^{n+1} (as found in Theorem 8.6). In particular, since the latter vanish, so do the former. $\square$

The Borel Conjecture posits that closed aspherical manifolds with isomorphic fundamental groups are necessarily homeomorphic. Our interest in contractible open manifolds led to the following.

Conjecture 7 (Weak Borel Conjecture)**.** *Closed aspherical manifolds with isomorphic fundamental groups have homeomorphic universal covers.*

Theorem 10.9 provides the means for a partial solution.

Theorem 10.10. *The Weak Borel Conjecture is true for those n-manifolds ($n \neq 4$) whose fundamental groups admits $\mathcal{Z}$-structures.*

Proof. Let P^n and Q^n be aspherical manifolds, and $\left(\overline{X}, Z\right)$ a $\mathcal{Z}$-structure on $\pi_1\left(P^n\right) \cong \pi_1\left(Q^n\right)$. By Bestvina's boundary swapping trick [Bes96, Lemma 1.4], both $\widetilde{P}^n$ and $\widetilde{Q}^n$ can be $\mathcal{Z}$-compactified by the addition of a copy of Z. Now apply Theorem 10.9. $\square$

Remark 19. Aspherical manifolds to which Theorem 10.10 applies include those with hyperbolic and CAT(0) fundamental groups. We are not aware of applications outside of those categories.

Recently, Bartels and Lück [BaLu12] proved the full-blown Borel Conjecture for δ-hyperbolic groups and CAT(0) groups that act geometrically on finite-dimensional CAT(0) spaces. Not surprisingly, their proof is more complicated than that of Theorem 10.10.

10.3. $\mathcal{EZ}$-structures in manifold topology. The notion of an $\mathcal{EZ}$-structure was formalized by Farrell and Lafont in [FaLa05]. Among its applications was a new proof of the Novikov Conjecture for δ-hyperbolic and CAT(0) groups. That result had been obtained earlier by Carlsson and Pedersen using similar ideas. We will not attempt to discuss the Novikov Conjecture here, except to say that it is related to, but much broader (and more difficult to explain) than the Borel Conjecture.

For a person with interests in manifold topology, one of the more intriguing aspects of Farrell and Lafont's work is a technique they develop which takes an arbitrary $\mathcal{Z}$-structures $\left(\overline{X}, Z\right)$ on a group G and replaces it with one of the form $(\mathbb{B}^n, Z)$, where n is necessarily large, Z is a topological copy of the original $\mathcal{Z}$-boundary lying in $\mathbb{S}^{n-1}$, and the new EG is the n-manifold with boundary $\mathbb{B}^n - Z$. The beauty here is

that, once the structure is established, all of the tools of high-dimensional manifold topology are available. In their introduction, they challenge the reader to find other applications of these manifold $\mathcal{Z}$-structures, likening them to the action of a Kleinian group on a compactified hyperbolic n-space.

Appendix A. Basics of ANR theory

Before beginning this appendix, we remind the reader that all spaces are separable metric.

A locally compact space X is an ANR (*absolute neighborhood retract*) if it can be embedded into $\mathbb{R}^n$ or, if necessary, $\mathbb{R}^\infty$ (a countable product of real lines) as a closed set in such a way that there exists a retraction $r : U \to X$, where U is a neighborhood of X. If the entire space $\mathbb{R}^n$ or $\mathbb{R}^\infty$ retracts onto X, we call X an AR (absolute retract). If X is finite-dimensional, all mention of $\mathbb{R}^\infty$ can be omitted. A finite-dimensional ANR is often called an ENR (*Euclidean neighborhood retract*) and a finite-dimensional AR an ER.

Use of the word "absolute" in ANR (or AR) stems from the following standard fact: If one embedding of X as a closed subset of $\mathbb{R}^n$ or $\mathbb{R}^\infty$ satisfies the defining condition, then so do all such embeddings. An alternative definition for ANR (and AR) is commonly found in the literature. To help avoid confusion, we offer that approach as Exercise A.3. Texts [Bor67] and [Hu65] are devoted entirely to the theory of ANRs; readers can go to either for details.

With a little effort (Exercise A.4) it can be shown that an AR is just a contractible ANR, so there is no loss of generality if focusing on ANRs.

A metric space Y is *locally contractible* if every ball $B(x;\varepsilon)$ contains a ball $B(x;\delta)$ that contracts within $B(x;\varepsilon)$. It is easy to see that every ANR is locally contractible. A partial converse gives a powerful characterization of finite-dimensional ANRs.

Theorem A.1. *A locally compact finite-dimensional space X is an ANR if and only if it is locally contractible.*

Example 39. *By Theorem A.1, manifolds, finite-dimensional locally finite polyhedra and CW complexes, and finite-dimensional proper CAT(0) spaces are all ANRs.*

Example 40. *It is also true that Hilbert cube manifolds, infinite-dimensional locally finite polyhedra and CW complexes, and infinite-dimensional proper CAT(0) spaces are all ANRs. Proofs require additional effort, but we will not hesitate to use these facts.*

Rather than listing key results individually, we provide a mix of facts about ANRs in a single Proposition. The first several are elementary, and the final item is a deep result. Each is an established part of ANR theory.

Proposition A.2 (Standard facts about ANRs)**.**

 (1) *Being an ANR is a local property: every open subset of an ANR is an ANR, and if every element of X has an ANR neighborhood, then X is an ANR.*

(2) *If $X = A \cup B$, where A, B, and $A \cap B$ are compact ANRs, then X is a compact ANR.*

(3) *Every retract of an ANR is an ANR; every retract of an AR is an AR.*

(4) *(Borsuk's Homotopy Extension Property) Every $h : (Z \times \{0\}) \cup (A \times [0,1]) \to X$, where A is closed in Z and X is ANR, admits an extension $H : Z \times [0,1] \to X$.*

(5) *(West, [Wes70]) Every ANR is proper homotopy equivalent to a locally finite CW complex; every compact ANR is homotopy equivalent to a finite complex.*

Remark 20. Items 4) and 5) allow us to extend the tools of algebraic topology and homotopy theory normally reserved for CW complexes to ANRs. For example, Whitehead's Theorem, that a map between CW complexes which induces isomorphisms on all homotopy groups is a homotopy equivalence, is also true for ANRs. In a very real sense, this sort of result is the motivation behind ANR theory.

Exercise A.3. *A locally compact space X is an ANE (absolute neighborhood extensor) if, for any space Z and any map $f : A \to X$, where A is a closed subset of Z, there is an extension $F : U \to X$ where U is a neighborhood of A. If an extension to all of Z is always possible, then X is an AE (absolute extensor). Show that being an ANE (or AE) is equivalent to being an ANR (or AR).* **Hint.** *The Tietze Extension Theorem will be helpful.*

Exercise A.4. *With the help of Exercise A.3 and the Homotopy Extension Property, prove that an ANR is an AR if and only if it is contractible.*

Exercise A.5. *A useful property of Euclidean space is that every compactum $A \subseteq \mathbb{R}^n$ has arbitrarily small compact polyhedral neighborhoods. Using the tools of Proposition A.2, prove the following CAT(0) analog: every compactum A in a proper CAT(0) space X has arbitrarily small compact ANR neighborhoods. (**Hint.** Cover A with compact metric balls.)*

Appendix B. Hilbert cube manifolds

This appendix is a very brief introduction to Hilbert cube manifolds. A primary goal is to persuade the uninitiated reader that there is nothing to fear. Although the main results from this area are remarkably strong (we sometimes refer to them as "Hilbert cube magic"), they are understandable and intuitive. Applying them is often quite easy.

The *Hilbert cube* is the infinite product $\mathcal{Q} = \prod_{i=1}^{\infty} [-1,1]$ with metric $d((x_i),(y_i)) = \sum \frac{|x_i - y_i|}{2^i}$. A *Hilbert cube manifold* is a separable metric space X with the property that each $x \in X$ has a neighborhood homeomorphic to $\mathcal{Q}$. Hilbert cube manifolds are interesting in their own right, but our primary interest stems from their usefulness in working with spaces that are not necessarily infinite-dimensional—often locally finite CW complexes or more general ANRs. Two classic examples where that approach proved useful are:

- Chapman [Cha74] used Hilbert cube manifolds to prove the topological invariance of Whitehead torsion for finite CW complexes, ie, homeomorphic finite complexes are simple homotopy equivalent.
- West [Wes70] used Hilbert cube manifolds to solve a problem of Borsuk, showing that every compact ANR is homotopy equivalent to a finite CW complex. (See Proposition A.2.)

The ability to attack a problem about ANRs using Hilbert cube manifolds can be largely explained using the following pair of results.

Theorem B.1 (Edwards, [Edw80])**.** *If A is an ANR, then $A \times \mathcal{Q}$ is a Hilbert cube manifold.*

Theorem B.2 (Triangulability of Hilbert Cube Manifolds, Chapman, [Cha76])**.** *If X is a Hilbert cube manifold, then there is a locally finite polyhedron K such that $X \approx K \times \mathcal{Q}$.*

A typical (albeit, simplified) strategy for solving a problem involving an ANR A might look like this:

A) Take the product of A with $\mathcal{Q}$ to get a Hilbert cube manifold $X = A \times \mathcal{Q}$.

B) Triangulate X, obtaining a polyhedron K with $X \approx K \times \mathcal{Q}$.

C) The polyhedral structure of K together with a variety of tools available in a Hilbert cube manifolds (see below) make solving the problem easier.

D) Return to A by collapsing out the $\mathcal{Q}$-factor in $X = A \times \mathcal{Q}$.

In these notes, most of our appeals to Hilbert cube manifold topology are of this general sort. That is not to say the strategy always works—the main result of [Gui01] (see Remark 15(a)) is one relevant example.

Tools available in a Hilbert cube manifold are not unlike those used in finite-dimensional manifold topology. We list a few such properties, without striving for best-possible results.

Proposition B.3 (Basic properties of Hilbert cube manifolds)**.** *Let X be a connected Hilbert cube manifold.*

(1) *(Homogeneity) For any pair $x_1, x_2 \in X$, there exists a homeomorphism $h : X \to X$ with $h(x_1) = x_2$.*

(2) *(General Position) Every map $f : P \to X$, where P is a finite polyhedron can be approximated arbitrarily closely by an embedding.*

(3) *(Regular Neighborhoods) Each compactum $C \subseteq X$ has arbitrarily small compact Hilbert cube manifold neighborhoods $N \subseteq X$. If C is a nicely embedded polyhedron, N can be chosen to strong deformation retract onto P.*

Exercise B.4. *As a special case, assertion (1) of Proposition B.3 implies that $\mathcal{Q}$ itself is homogeneous. This remarkable fact is not hard to prove. A good start is to construct a homeomorphism $h : \mathcal{Q} \to \mathcal{Q}$ with $h(1, 1, 1, \cdots) = (0, 0, 0, \cdots)$. To begin, think of a homeomorphism $k : [-1, 1] \times [-1, 1]$ taking $(1, 1)$ to $(0, 1)$, and use it to obtain $h_1 : \mathcal{Q} \to \mathcal{Q}$ with $h_1(1, 1, 1, \cdots) = (0, 1, 1, \cdots)$. Complete this argument.*

Example 41. *Here is another special case worth noting. Let K be an arbitrary locally finite polyhedron—for example, a graph. Then $K \times \mathcal{Q}$ is homogeneous.*

The material presented here is just a quick snapshot of the elegant and surprising world of Hilbert cube manifolds. A brief and readable introduction can be found in [Cha76]. Just for fun, we close by stating two more remarkable theorems that that are emblematic of the subject.

Theorem B.5 (Toruńczyk, [Tor80])**.** *An ANR X is a Hilbert cube manifold if and only if it satisfies the General Position property (Assertion (2)) of Proposition B.3.*

Theorem B.6 (Chapman, [Cha76])**.** *A map $f : K \to L$ between locally finite polyhedra is an (infinite) simple homotopy equivalence if and only if $f \times \mathrm{id}_{\mathcal{Q}} : K \times \mathcal{Q} \to L \times \mathcal{Q}$ is (proper) homotopic to a homeomorphism.*

References

[ADG97] F.D. Ancel, M.W. Davis, C.R. Guilbault, *CAT(0) reflection manifolds.* Geometric Topology (Athens, GA, 1993), 441–445, AMS/IP Stud. Adv. Math., 2.1, Amer. Math. Soc., Providence, RI, 1997.

[AnGu99] F.D. Ancel, C.R. Guilbault, *$\mathcal{Z}$-compactifications of open manifolds.* Topology 38 (1999), no. 6, 1265–1280.

[AnSi85] F.D. Ancel and L.C. Siebenmann, *The construction of homogeneous homology manifolds*, Abstracts Amer. Math. Soc. 6 (1985), abstract 816-57-72.

[BaLu12] A. Bartels, W. Lück, *The Borel Conjecture for hyperbolic and CAT(0)-groups* Ann. of Math. (2) 175 (2012), 631-689.

[Ben92] N. Benakli, *Polyedres hyperboliques, passage du local au global*, PhD Thesis, University d'Orsay, 1992.

[Bel] I. Belegradek, *Open aspherical manifolds not covered by the Euclidean space.* arXiv: 1208.5666.

[Bes96] M. Bestvina, *Local homology properties of boundaries of groups.* Michigan Math. J. 43 (1996), no. 1, 123–139.

[BeMe91] M. Bestvina, G. Mess, *The boundary of negatively curved groups.* J. Amer. Math. Soc. 4 (1991), no. 3, 469–481.

[Bor67] K. Borsuk, *Theory of retracts.* Monografie Matematyczne, Tom 44, Warsaw, 1967.

[Bor75] K. Borsuk, *Theory of Shape*, Monografie Matematyczne Tom 59,Warszawa, 1975.

[Bow98] B.H. Bowditch, *Cut points and canonical splittings of hyperbolic groups.* Acta Math. 180 (1998), no. 2, 145–186.

[BLL65] W. Browder, J. Levine, G.R. Livesay, *Finding a boundary for an open manifold*, Amer. J. Math. 87 (1965) 1017–1028.

[Bro94] K.S. Brown, *Cohomology of groups.* Corrected reprint of the 1982 original. Graduate Texts in Mathematics, 87. Springer-Verlag, New York, 1994. x+306 pp.

[CaPe95] G. Carlsson, E.K. Pedersen, *Controlled algebra and the Novikov conjectures for K-and L-theory*, Topology 34 (1995) 731–758.

[CaJu94] A. Casson, D. Jungreis, *Convergence groups and Seifert fibered 3-manifolds*, Invent. Math. 118 (1994), no. 3, 441–456.

[Cha74] T.A. Chapman, *Topological invariance of Whitehead torsion.* Amer. J. Math. 96 (1974), 488–497.

[Cha76] T.A. Chapman, *Lectures on Hilbert cube manifolds.* Expository lectures from the CBMS Regional Conference held at Guilford College, October 11-15, 1975. Regional Conference Series in Mathematics, No. 28. American Mathematical Society, Providence, R. I., 1976. x+131 pp.

[ChSi76] T. A. Chapman, L.C. Siebenmann, *Finding a boundary for a Hilbert cube manifold.* Acta Math. 137 (1976), no. 3-4, 171–208.

[ChKi70] J. Cheeger, J.M. Kister, *Counting topological manifolds.* Topology 9 1970 149–151.

[CrKl00] C.B. Croke, B. Kleiner, *Spaces with nonpositive curvature and their ideal boundaries.* Topology 39 (2000), no. 3, 549–556.

[Dah03] F. Dahmani, *Classifying spaces and boundaries for relatively hyperbolic groups*, Proc. London Math. Soc. (3) 86 (2003) 666–684.

[Dvr86] R.J. Daverman, *Decompositions of manifolds.* Pure and Applied Mathematics, 124. Academic Press, Inc., Orlando, FL, 1986. xii+317 pp.

[DaTi96] R.J. Daverman, F.C. Tinsley, *Controls on the plus construction.* Michigan Math. J. 43 (1996), no. 2, 389–416.

[Dvs83] M.W. Davis, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space.* Ann. of Math. (2) 117 (1983), no. 2, 293–324.

[DaJa91] M.W. Davis, T. Januszkiewicz, *Hyperbolization of polyhedra.* J. Differential Geom. 34 (1991), no. 2, 347–388.

[Dra99] A.N. Dranishnikov, *Boundaries of Coxeter groups and simplicial complexes with given links*, J. Pure Appl. Algebra 137 (1999), no. 2, 139-151.

[Dra06] A.N. Dranishnikov, *On Bestvina-Mess formula.* Topological and asymptotic aspects of group theory, 77–85, Contemp. Math., 394, Amer. Math. Soc., Providence, RI, 2006.

[DySe78] J. Dydak, J. Segal, *Shape theory. An introduction.* Lecture Notes in Mathematics, 688. Springer, Berlin, 1978. vi+150 pp.

[EdHa76] D.A. Edwards, H.M. Hastings, *Every weak proper homotopy equivalence is weakly properly homotopic to a proper homotopy equivalence.* Trans. Amer. Math. Soc. 221 (1976), no. 1, 239–248.

[Edw80] R.D. Edwards, *Characterizing infinite-dimensional manifolds topologically (after Henryk Toruńczyk).* Séminaire Bourbaki (1978/79), Exp. No. 540, pp. 278–302, Lecture Notes in Math., 770, Springer, Berlin-New York, 1980.

[FaLa05] F.T. Farrell, J-F Lafont, *EZ-structures and topological applications*, Comment. Math. Helv. 80 (2005) 103–121.

[Fer00] S.C. Ferry, *Stable compactifications of polyhedra.* Michigan Math. J. 47 (2000), no. 2, 287–294.

[Fer] S.C. Ferry, *Geometric Topology Notes,* book in progress.

[FiGu05] H. Fischer, C.R. Guilbault, *On the fundamental groups of trees of manifolds.* Pacific J. Math. 221 (2005), no. 1, 49–79.

[Fre95] E.M. Freden, *Negatively curved groups have the convergence property. I*, Ann. Acad. Sci.Fenn. Ser. A I Math. 20 (1995), no. 2, 333–348.

[FrQu90] M.H. Freedman, F. Quinn, *Topology of 4-manifolds*, Princeton University Press, Princeton, New Jersey, 1990.

[Gab92] D. Gabai, *Convergence groups are Fuchsian groups*, Ann. of Math. (2) 136 (1992), no. 3, 447-510.

[Geo86] R. Geoghegan, *The shape of a group—connections between shape theory and the homology of groups.* Geometric and algebraic topology, 271–280, Banach Center Publ., 18, PWN, Warsaw, 1986.

[Geo08] R. Geoghegan, *Topological methods in group theory.* Graduate Texts in Mathematics, 243. Springer, New York, 2008. xiv+473 pp.

[GeMi85] R. Geoghegan, M.L. Mihalik, *Free abelian cohomology of groups and ends of universal covers.* J. Pure Appl. Algebra 36 (1985), no. 2, 123–137.

[Gro87] M. Gromov, *Hyperbolic groups*, S.M. Gersten (ed.) , Essays in Group Theory , MSRI Publ., 8 , Springer (1987) pp. 75–263.

[Gui00] C. R. Guilbault, *Manifolds with non-stable fundamental groups at infinity.* Geom. Topol. 4 (2000), 537–579.

[Gui01] C. R. Guilbault, *A non-Z-compactifiable polyhedron whose product with the Hilbert cube is Z-compactifiable.* Fund. Math. 168 (2001), no. 2, 165–197.

[Gui07] C. R. Guilbault, *Products of open manifolds with* ℝ. Fund. Math. 197 (2007), 197–214.

[GuTi03] C.R. Guilbault, F.C. Tinsley, *Manifolds with non-stable fundamental groups at infinity, II*, Geometry and Topology, 7(2003) 255–286.

[GuTi06] C.R. Guilbault, F.C. Tinsley, *Manifolds with non-stable fundamental groups at infinity, III*, Geometry and Topology, 10(2006) 541-556.

[GuTi] C.R. Guillbault, F.C. Tinsley, *Manifolds that are inward tame at infinity*, in progress.

[Hu65] S-t. Hu, *Theory of retracts.* Wayne State University Press, Detroit 1965 234 pp.

[HuRa96] B. Hughes, A. Ranicki, *Ends of complexes.* Cambridge Tracts in Mathematics, 123. Cambridge University Press, Cambridge, 1996. xxvi+353 pp.

[JaSw06] T. Januszkiewicz, J. Świątkowski, *Simplicial nonpositive curvature.* Publ. Math. Inst. Hautes Études Sci. No. 104 (2006), 1–85.

[KaBe02] I. Kapovich, N. Benakli, *Boundaries of hyperbolic groups.* Combinatorial and geometric group theory (New York, 2000/Hoboken, NJ, 2001), 39–93, Contemp. Math., 296, Amer. Math. Soc., Providence, RI, 2002.

[Ker69] M.A. Kervaire, *Smooth homology spheres and their fundamental groups.* Trans. Amer. Math. Soc. 144 (1969) 67–72.

[Mat65] M. Mather, *Counting homotopy types of manifolds.* Topology 3 (1965) 93–94.

[Maz61] B. Mazur, *A note on some contractible 4-manifolds.* Ann. of Math. (2) 73 (1961) 221–228.

[May07] M.C. May, *Finite dimensional Z-compactifications.* Thesis (Ph.D.)–The University of Wisconsin - Milwaukee. 2007.

[Mc62] D.R. McMillan, Jr. *Some contractible open 3-manifolds.* Trans. Amer. Math. Soc. 102 (1962) 373–382.

[McTh80] D.R. McMillan, T.L. Thickstun, *Open three-manifolds and the Poincaré conjecture.* Topology 19 (1980), no. 3, 313–320.

[Mih83] M.L. Mihalik, *Semistability at the end of a group extension.* Trans. Amer. Math. Soc. 277 (1983), no. 1, 307–321.

[Mih87] M.L. Mihalik, *Senistability at* ∞, *∞-ended groups and group cohomology.* Trans. Amer. Math. Soc. 303 (1987), no. 2, 479–485.

[Mih96] M.L. Mihalik, *Semistability of Artin and Coxeter groups.* J. Pure Appl. Algebra 111 (1996), no. 1-3, 205–211.

[MiT92a] M.L. Mihalik, S.T. Tschantz, *One relator groups are semistable at infinity.* Topology 31 (1992), no. 4, 801–804.

[MiT92b] M.L. Mihalik, S.T. Tschantz, *Semistability of amalgamated products and HNN-extensions.* Mem. Amer. Math. Soc. 98 (1992), no. 471, vi+86 pp.

[Moo08] C.P. Mooney, *Examples of non-rigid CAT(0) groups from the category of knot groups.* Algebr. Geom. Topol. 8 (2008), no. 3, 1666–1689.

[Moo10] C.P. Mooney, *Generalizing the Croke-Kleiner construction.* Topology Appl. 157 (2010), no. 7, 1168–1181.

[Mye88] R. Myers, *Contractible open 3-manifolds which are not covering spaces.* Topology 27 (1988), no. 1, 27–35.

[New48] M. H. A. Newman, *Boundaries of ULC sets in Euclidean n-space.* Proc. Nat. Acad. Sci. U.S.A. 34 (1948), 193–196

[Ont05] P. Ontaneda, *Cocompact CAT(0) spaces are almost geodesically complete.* Topology 44 (2005), no. 1, 47–62.

[OsPr09] D. Osajda, P. Przytycki, *Boundaries of systolic groups.* Geom. Topol. 13 (2009), no. 5, 2807–2880.

[PaSw09] P. Papasoglu, E. Swenson, *Boundaries and JSJ decompositions of CAT(0)-groups.* Geom. Funct. Anal. 19 (2009), no. 2, 559–590.

[Qui79] F. Quinn, *Ends of maps. I.* Ann. of Math. (2) 110 (1979), no. 2, 275–331.

[Qui88] F. Quinn, *Homotopically stratified sets.* J. Amer. Math. Soc. 1 (1988), no. 2, 441–499.

[RoSa82] C.P. Rourke and B.J. Sanderson, *Introduction to piecewise-linear topology*, Reprint. Springer Study Edition. Springer-Verlag, Berlin-New York, 1982. viii+123 pp.

[Rua06] K. Ruane, *CAT(0) groups with specified boundary.* Algebr. Geom. Topol. 6 (2006), 633–649 (electronic).

[ScWa79] P. Scott, T. Wall, *Topological methods in group theory.* Homological group theory (Proc. Sympos., Durham, 1977), pp. 137–203, London Math. Soc. Lecture Note Ser., 36, Cambridge Univ. Press, Cambridge-New York, 1979.

[Sie65] L.C. Siebenmann, *The obstruction to finding a boundary for an open manifold of dimension greater than five*, Ph.D. thesis, Princeton University, 1965.

[Sie70] L.C. Siebenmann, *Infinite simple homotopy types.* Indag. Math. 32 (1970), 479–495.

[Sta62] J. Stallings, *The piecewise-linear structure of Euclidean space.* Proc. Cambridge Philos. Soc. 58 (1962), 481–488.

[Sta68] J. Stallings, *On torsion-free groups with infinitely many ends.* Ann. of Math. (2) 88 (1968) 312–334.

[Swa96] A. Swarup, *On the cut point conjecture*, Electron. Res. Announc. Amer. Math. Soc. 2 (1996), 98–100.

[Swe99] E.L. Swenson, *A cut point theorem for CAT(0) groups.* J. Differential Geom. 53 (1999), no. 2, 327–358.

[Tir11] C.J. Tirel, *$\mathcal{Z}$-structures on product groups.* Algebr. Geom. Topol. 11 (2011), no. 5, 2587–2625.

[Tor80] H. Toruńczyk, *On CE-images of the Hilbert cube and characterization of Q-manifolds.* Fund. Math. 106 (1980), no. 1, 31–40.

[Tuc74] T.W. Tucker, *Non-compact 3-manifolds and the missing-boundary problem.* Topology 13 (1974), 267–273.

[Tuk88] P. Tukia, *Homeomorphic conjugates of Fuchsian groups*, J. Reine Angew. Math. 391 (1988),1-54.

[Wal65] C. T. C. Wall, *Finiteness conditions for CW-complexes.* Ann. of Math. (2) 81 1965 56–69.

[Wes70] J.E. West, *Mapping Hilbert cube manifolds to ANR's: a solution of a conjecture of Borsuk.* Ann. of Math. (2) 106 (1977), no. 1, 1–18.

[Whi35] J. H.C. Whitehead, *A certain open manifold whose group is unity*, Quarterly journal of Mathematics 6 (1) (1935) 268–279.

[Wil05] J.M. Wilson, *A CAT(0) group with uncountably many distinct boundaries.* J. Group Theory 8 (2005), no. 2, 229–238.

Department of Mathematical Sciences, University of Wisconsin-Milwaukee, Milwaukee, Wisconsin 53201

E-mail address: craigg@uwm.edu